PRAYAS 2.0

FOR IIT - JEE 2023

COORDINATE GEOMETRY

HYPERBOLA

LEC – 01

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TODAY's GOAL

Properties / Highlights of Ellipse

HYPERBOLA

Equation of Standard Hyperbola

Basic Terminology

OP-QP

LAST CLASS

Four Important Terms:

C.O.C. $\quad$ C.W.G.M.P. $\quad$ P.O.T. $\quad$ P&P

$T_1 = 0 \quad T_1 = S_1 \quad T_1^2 = SS_1 \quad T_1 = 0$

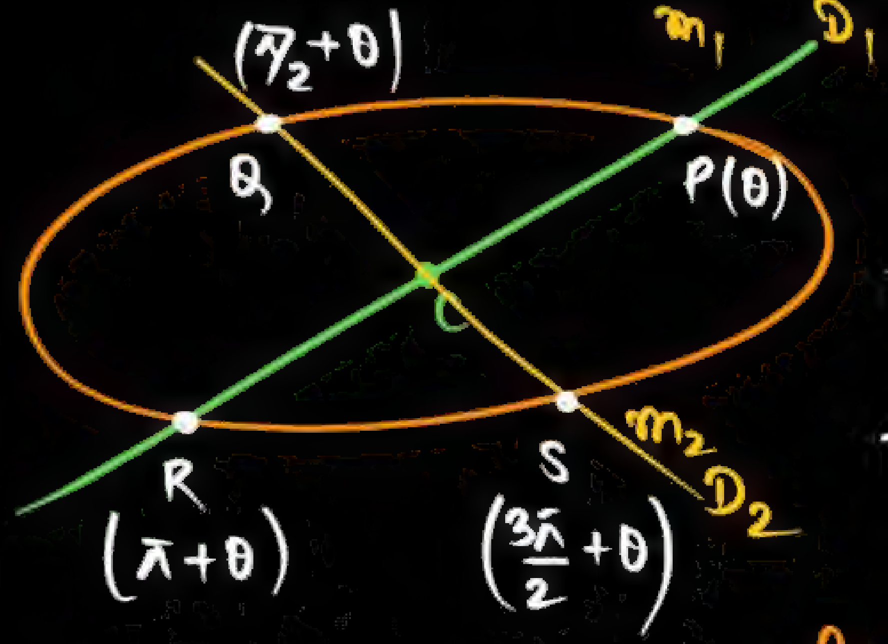
Diameter & Conjugate Diameter:

$y = \dfrac{-b^2}{a^2 m} x$

slopes (m_1 & m_2)

$\Downarrow$

$m_1 m_2 = \dfrac{-b^2}{a^2}$



$CP^2 + CQ^2 = a^2 + b^2$

area (||gm) = 4ab

formed by Tangents at P, Q, R & S.

Q. Let $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$. Let E_2 be another ellipse such that it touches the end points of major axis of E_1 and the foci of E_2 are the end points of minor of E_1. If E_1 and E_2 have same eccentricities, then its value is

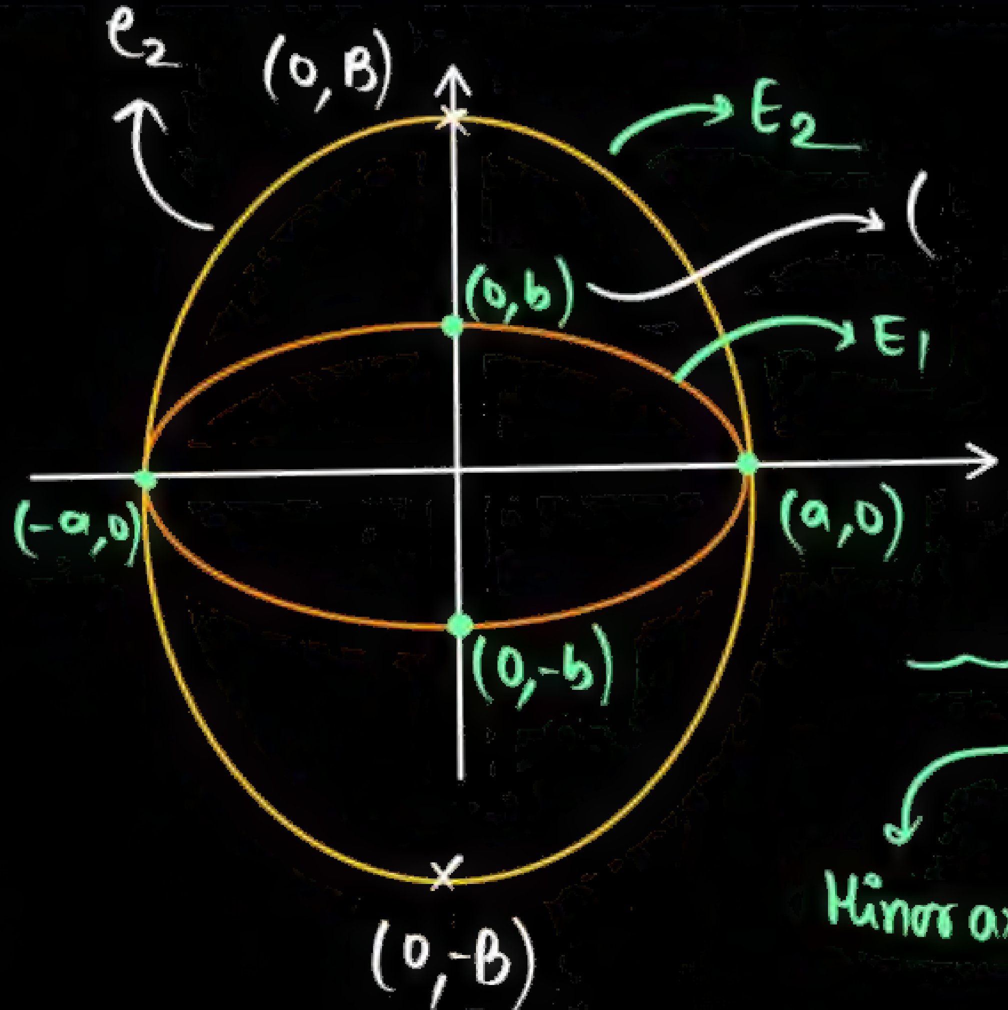
[JEE Mains-2021]

A. $\frac{-1+\sqrt{5}}{2}$ ✓

B. $\frac{-1+\sqrt{8}}{2}$

C. $\frac{-1+\sqrt{3}}{2}$

D. $\frac{-1+\sqrt{6}}{2}$



e_2^2

$e_1^2 = 1 - \frac{b^2}{a^2}$

$E_2 : \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

Minor axis $= 2a$ $\boxed{A = a}$

$a = A$

$b = B e_2$

$e_2^2 = 1 - \frac{A^2}{B^2}$

$e_2^2 = 1 - \frac{a^2}{(b/e_2)^2}$

$e_2 = \frac{\sqrt{5}-1}{2} \Leftarrow e_2 = \sqrt{\frac{6-2\sqrt{5}}{4}}$

$(\sqrt{5}-1)^2$

$$e_2^2 = 1 - \frac{a^2}{b^2}(e_2^2)$$

$$e_2^2 + \frac{a^2}{b^2}e_2^2 = 1$$

$$e_2^2\left(1+\frac{a^2}{b^2}\right) = 1.$$

$$e_2^2\left(1+\frac{1}{1-e_2^2}\right) = 1.$$

$$e_2^2(1-e_2^2+1) = 1-e_2^2$$

$$e_1^2 = e_2^2 = 1-\frac{b^2}{a^2}$$

$$\frac{b^2}{a^2} = 1-e_2^2$$

$$e_2^2 = \alpha$$

$$\alpha(2-\alpha) = 1-\alpha$$

$$2\alpha - \alpha^2 = 1-\alpha$$

$$\alpha^2 - 3\alpha + 1 = 0.$$

$$\alpha = e_2^2 = \frac{3\pm\sqrt{5}}{2}$$

$$e_2 = \sqrt{\frac{3-\sqrt{5}}{2}}$$

Q. If a tangent of slope $\frac{1}{3}$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$ is normal to the circle $x^2+y^2+2x+2y+1=0$

$(-1,-1)$

(A) maximum value of ab is $\frac{2}{3}$

(B) $a \in \left(\sqrt{\frac{2}{5}}, 2\right)$

(C) $a \in \left(\frac{2}{3}, 2\right)$

(D) maximum value of ab is 1

$m=\frac{1}{3}$

$\# \quad y=\frac{1}{3}x \pm \sqrt{\frac{a^2}{9}+b^2}$

$(-1,-1)$

$-1=-\frac{1}{3} \pm \sqrt{\frac{a^2}{9}+b^2}$

$\mp\frac{2}{3}=\pm\sqrt{\frac{a^2}{9}+b^2}$

Sq

$\boxed{\frac{4}{9}=\frac{a^2}{9}+b^2}$

$\div a^2$

$\frac{4}{9a^2}=\frac{1}{9}+\frac{b^2}{a^2}$

$e^2=1-\frac{b^2}{a^2}$

$e^2=1-\left(\frac{4}{9a^2}-\frac{1}{9}\right)$

$e^2=\frac{10}{9}-\frac{4}{9a^2}<1$

$\# \quad \frac{a^2}{9}, b^2 \rightarrow AM \geq GM$

$\Downarrow$

$\frac{\frac{a^2}{9}+b^2}{2} \geq \sqrt{\frac{a^2b^2}{9}}$

$\frac{4}{3\cancel{9}(2)} \geq \frac{ab}{\cancel{3}}$

$\boxed{\frac{2}{3} \geq ab}$

$$0 < \frac{10}{9} - \frac{4}{9a^2} < 1$$

\# $a, b \equiv +ve.$

$$\frac{4}{\cancel{9}a^2} < \frac{10}{\cancel{9}}$$

$$\frac{4}{10} < a^2$$

$$\Downarrow$$

$$\frac{2}{5} < a^2$$

$$a^2 - \frac{2}{5} > 0$$

$$\left(a - \sqrt{\frac{2}{5}}\right)\left(a + \sqrt{\frac{2}{3}}\right) > 0$$

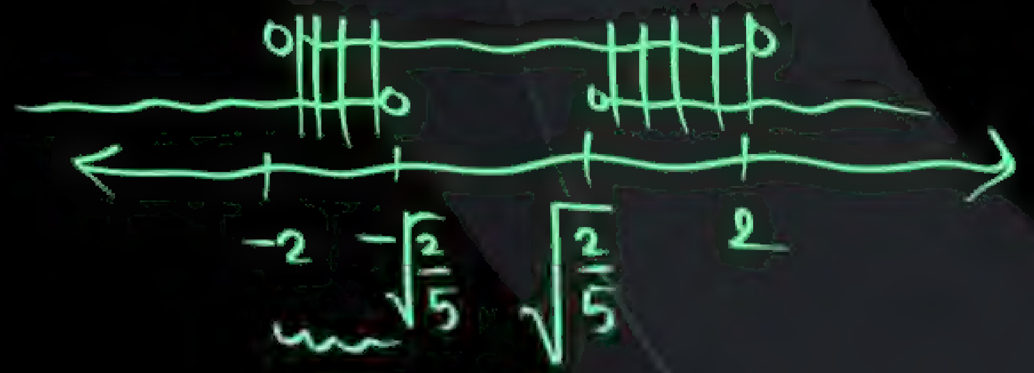
$$\frac{10}{9} - 1 < \frac{4}{9a^2}$$

$$\frac{1}{\cancel{9}} < \frac{4}{\cancel{9}a^2}$$

$$\boxed{a^2 < 4}$$

$$a^2 - 4 < 0$$

$$(a-2)(a+2) < 0$$



$$\Downarrow$$

\# $a \in \left(\sqrt{\frac{2}{5}}, 2\right)$

Q. If two concentric ellipses be such that the foci of one be on the other and if $\frac{\sqrt{3}}{2}$ and $\frac{1}{\sqrt{2}}$ be their eccentricities. Then angle between their axes is

CHALLENGER

A. $\cos^{-1}\sqrt{\frac{2}{3}}$ ✓

B. $\cos^{-1}\frac{2}{3\sqrt{3}}$

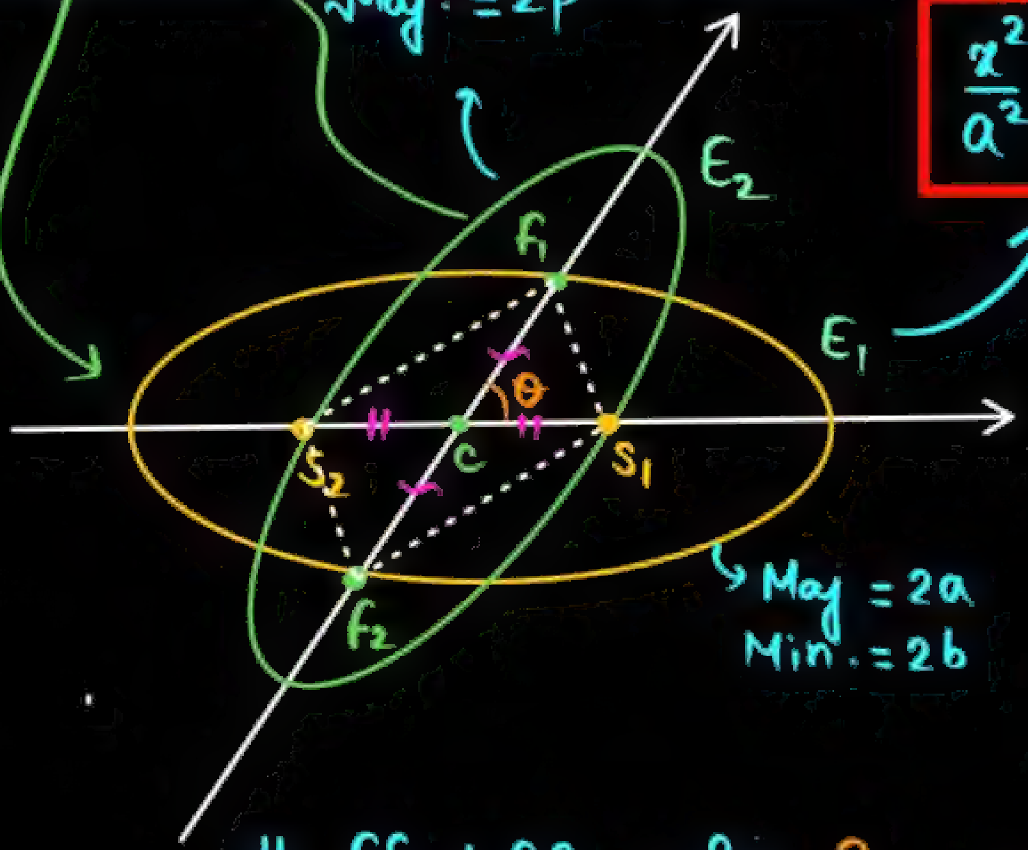
C. $\cos^{-1}\frac{1}{\sqrt{6}}$

D. $\cos^{-1}\frac{\sqrt{2}}{3}$

Min. $= 2q$

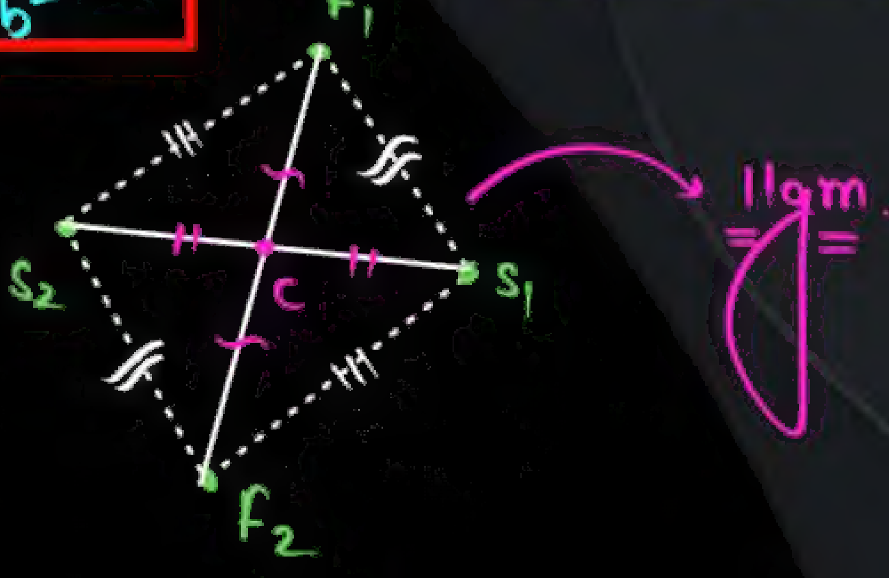
Maj. $= 2p$

$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$



Maj $= 2a$

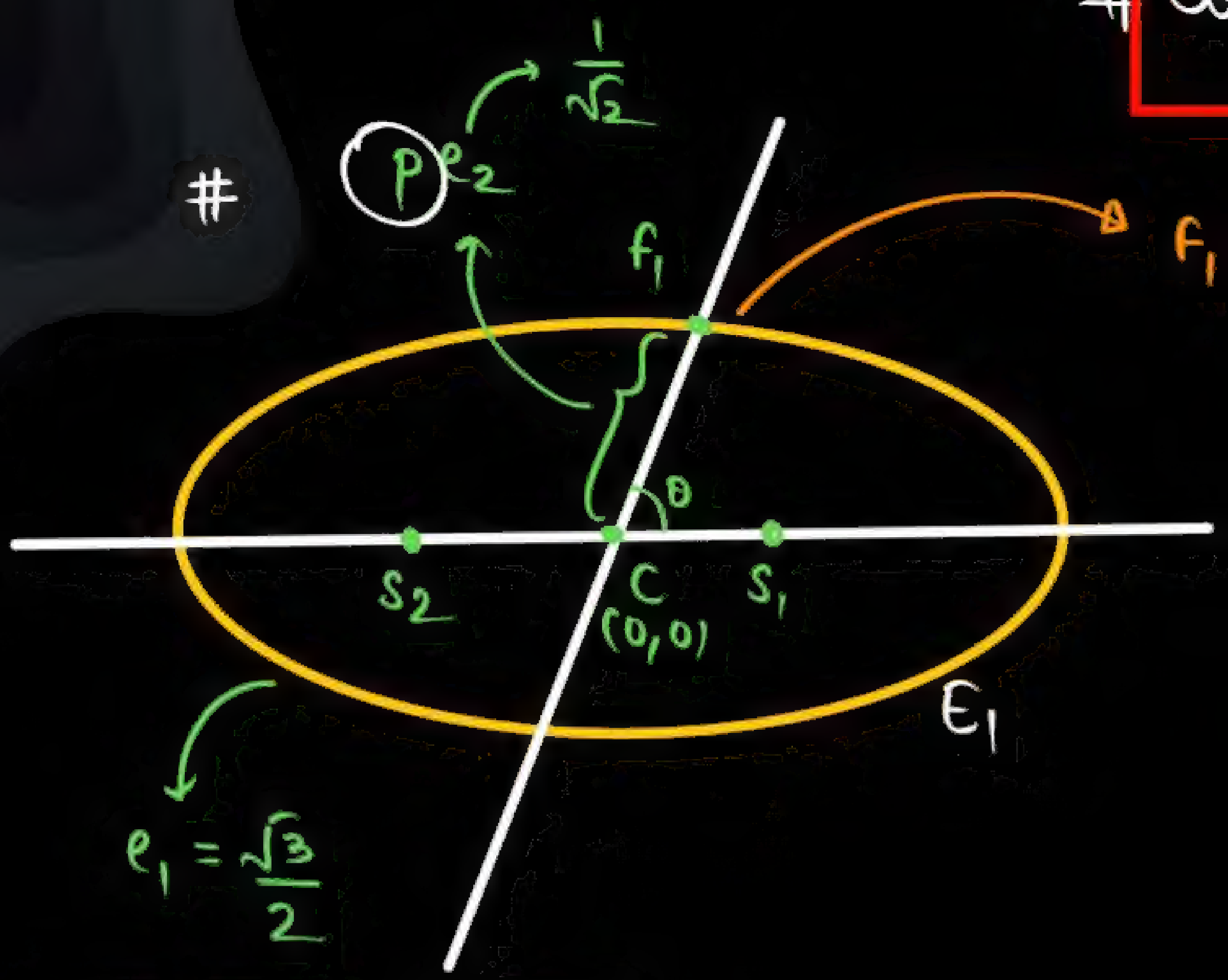
Min. $= 2b$



llgm.

$f_1S_1 + f_1S_2 = 2a$
$F_1S_1 + S_1F_2 = 2p$

$\Rightarrow \boxed{a = p}$ ✓



$\boxed{\cos^2\theta = \frac{2}{3}} \Leftarrow +\frac{3\cos^2\theta}{2} = +1$

$\frac{\cos^2\theta}{2} + 2 - 2\cos^2\theta = 1$

$f_1 \equiv (\alpha, \beta)$

$\frac{\alpha - 0}{\cos\theta} = \frac{\beta - 0}{\sin\theta} = +pe_2$

$f_1 \begin{cases} \alpha = pe_2\cos\theta \\ \beta = pe_2\sin\theta \end{cases}$

Put f_1

Lies on $E_1 \equiv \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

$\frac{\cancel{p^2} e_2^2\cos^2\theta}{\cancel{a^2}} + \frac{p^2 e_2^2\sin^2\theta}{b^2} = 1$

$\frac{a^2}{b^2}$

$\frac{1}{2}\cos^2\theta + 4\left(\frac{1}{2}\right)\sin^2\theta = 1 \Rightarrow \frac{\cos^2\theta}{2} + 2(1-\cos^2\theta) = 1$

$e_1 = \frac{\sqrt{3}}{2} \Rightarrow e_1^2 = 1 - \frac{b^2}{a^2}$

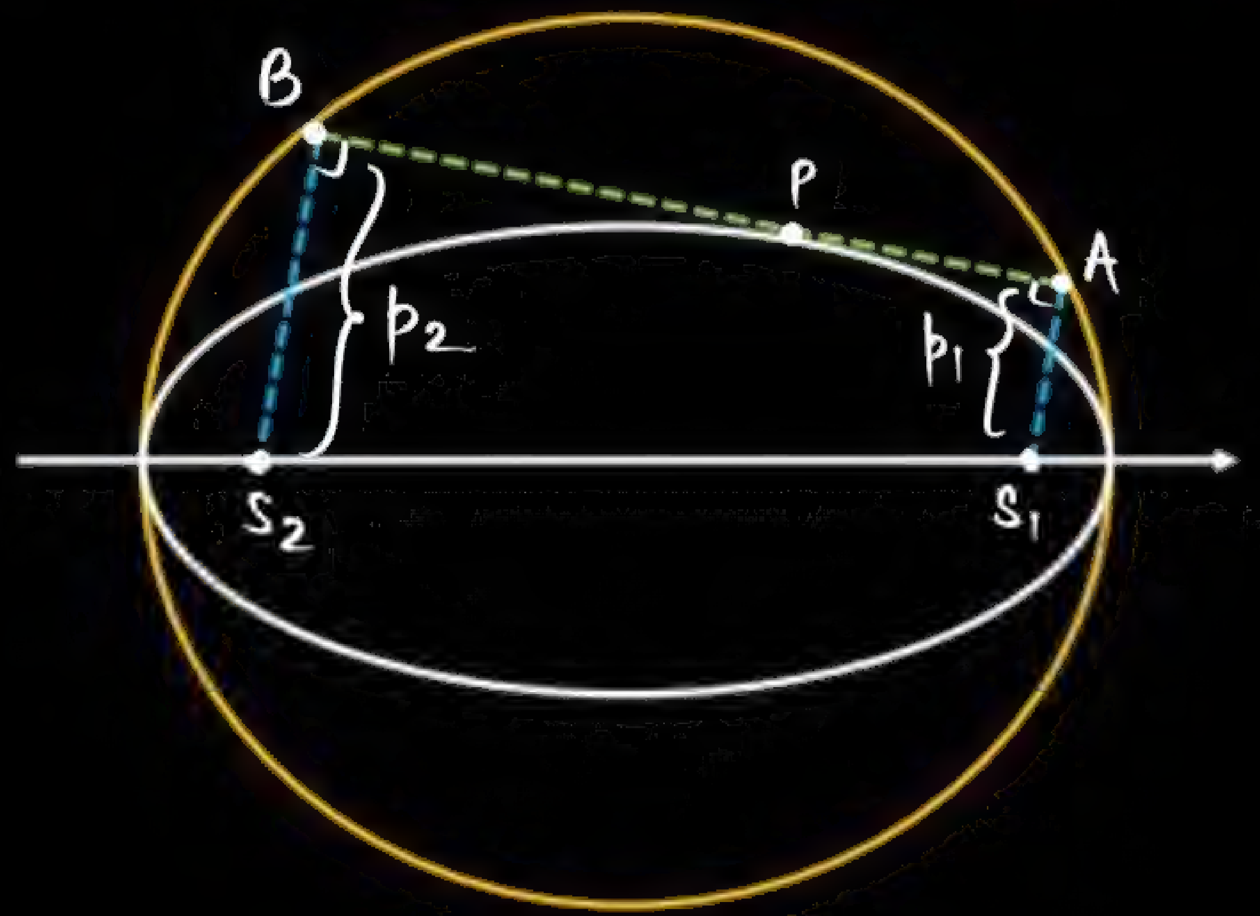
$e_2 = \frac{1}{\sqrt{2}}$ $\qquad \frac{b^2}{a^2} = 1 - \frac{3}{4} = \frac{1}{4}$

$p = a$ $\qquad \frac{b^2}{a^2} = \frac{1}{4} \Rightarrow \boxed{\frac{a^2}{b^2} = 4}$

PROPERTIES OF ELLIPSE

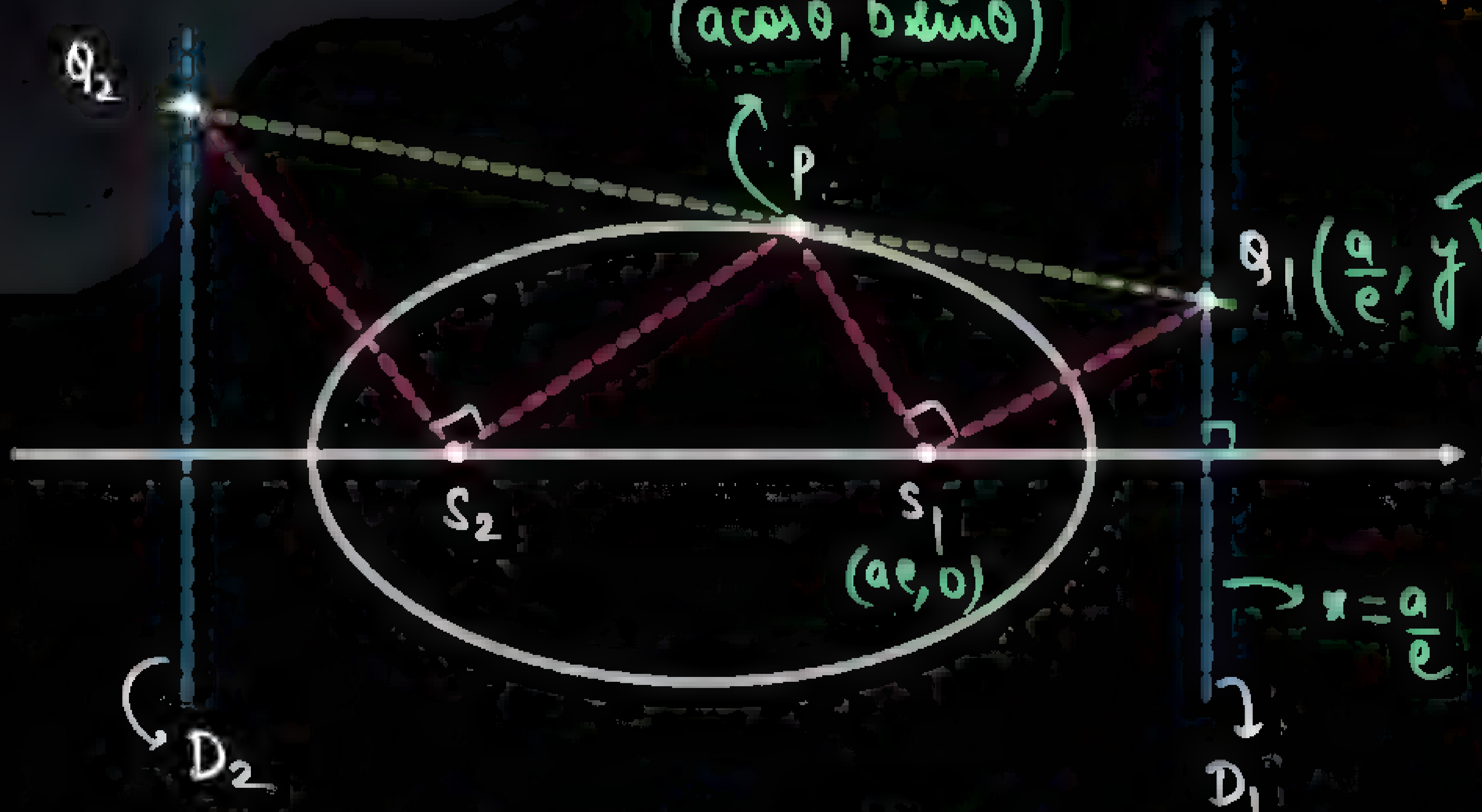
P-1: Locus of foot of perpendicular drawn from foci on any tangent is Auxiliary Circle.

P-2: Product of lengths of perpendiculars from foci on Tangent is always constant & equals to (semi-minor axis)^2



$p_1 p_2 = (\text{semi-minor axis})^2$

P-3: Portion of tangent intercepted between point of contact and directrix subtend 90° at corresponding focus.



Proof :

$$T_P : \boxed{\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1}$$

$$x = \frac{a}{e}$$

$$\frac{\cancel{a}}{e}\,\frac{\cos\theta}{\cancel{a}} + \frac{y\sin\theta}{b} = 1$$

$$\frac{y\sin\theta}{b} = 1 - \frac{\cos\theta}{e}$$

$$y = \frac{b}{\sin\theta}\left(\frac{e-\cos\theta}{e}\right)$$

$$m_{PS_1} = \frac{b\sin\theta}{a\cos\theta - ae}$$

$$m_{PS_1} = \frac{b}{a}\,\frac{\sin\theta}{(\cos\theta - e)}$$

$$m_{S_1Q_1} = \frac{y}{\frac{a}{e} - ae}$$

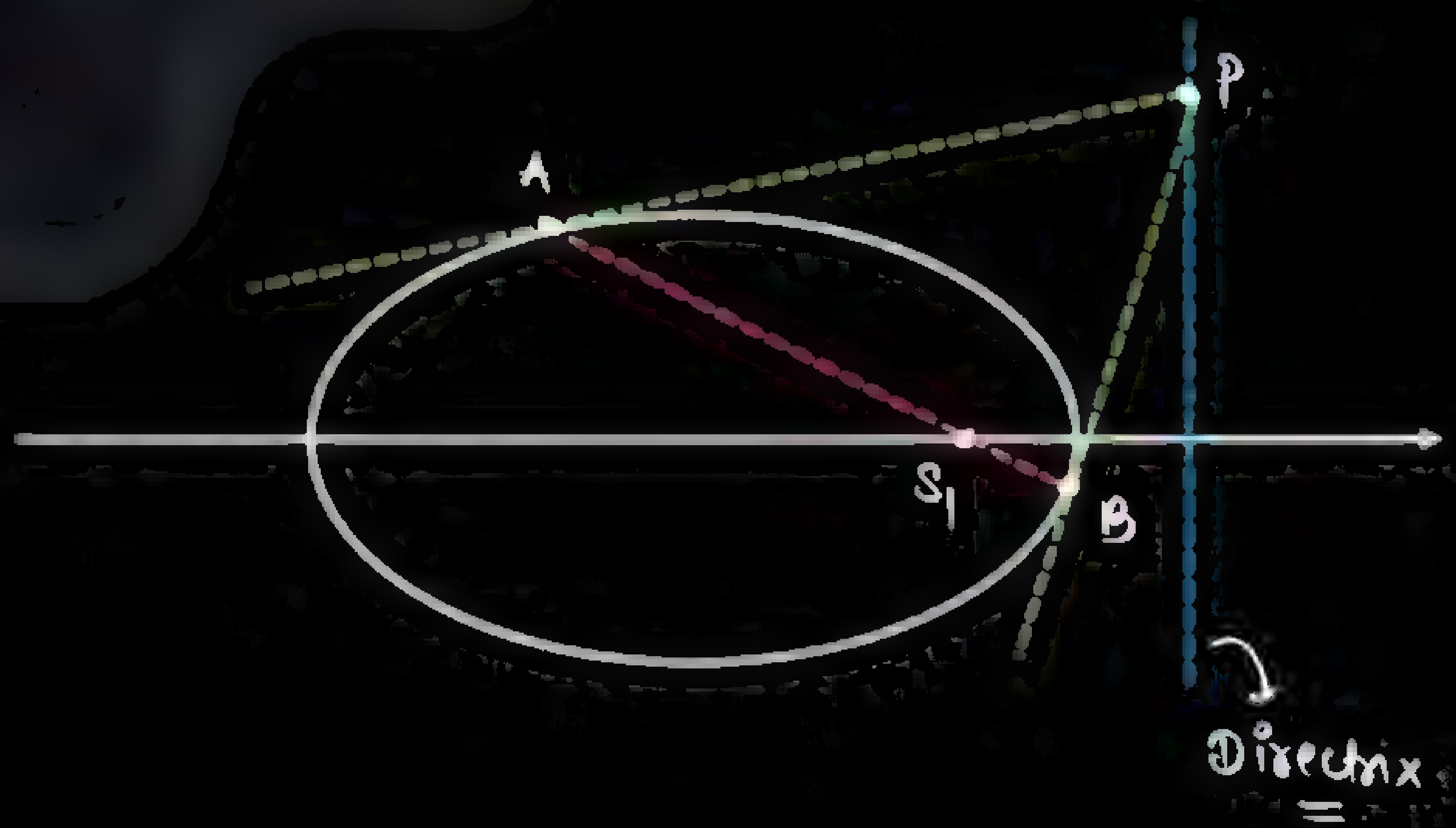
$$\times \Rightarrow \frac{b\sin\theta}{a(\cos\theta - e)} \times \frac{ye}{a(1-e^2)}$$

$$-1 = \frac{b^2}{-a^2(1-e^2)} \Leftarrow \frac{b\cancel{\sin\theta}}{a\cancel{(\cos\theta - e)}} \times \frac{\cancel{e}}{a(1-e^2)} \times \frac{b}{\cancel{\sin\theta}} \times \frac{\cancel{(e-\cos\theta)}}{\cancel{e}}$$

-1

b^2/a^2

P-4: Chord of contact corresponding to any point on directrix always passes through corresponding focus.



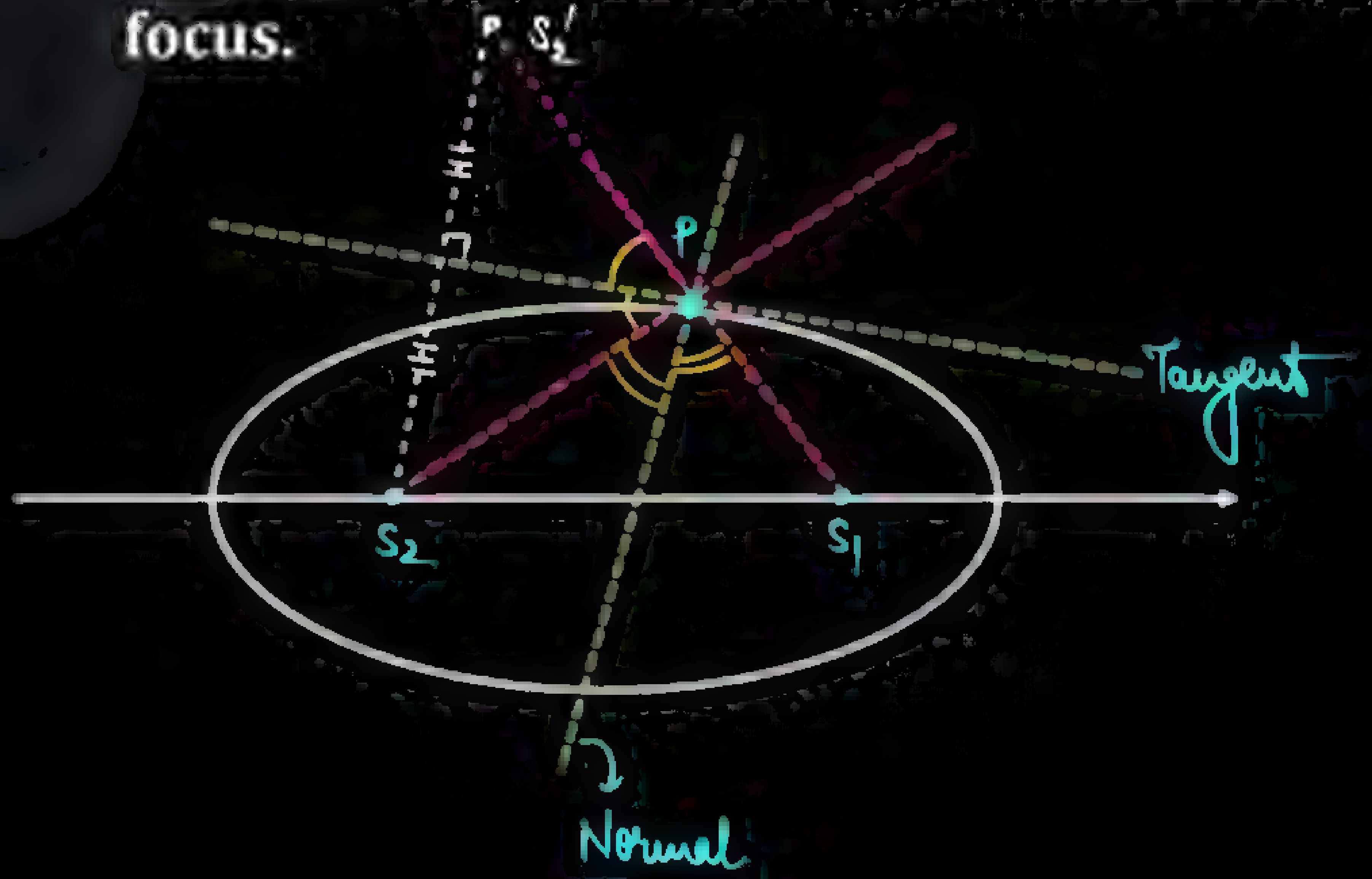
Note: *If focus is Pole then Directrix is Polar.*

(i)

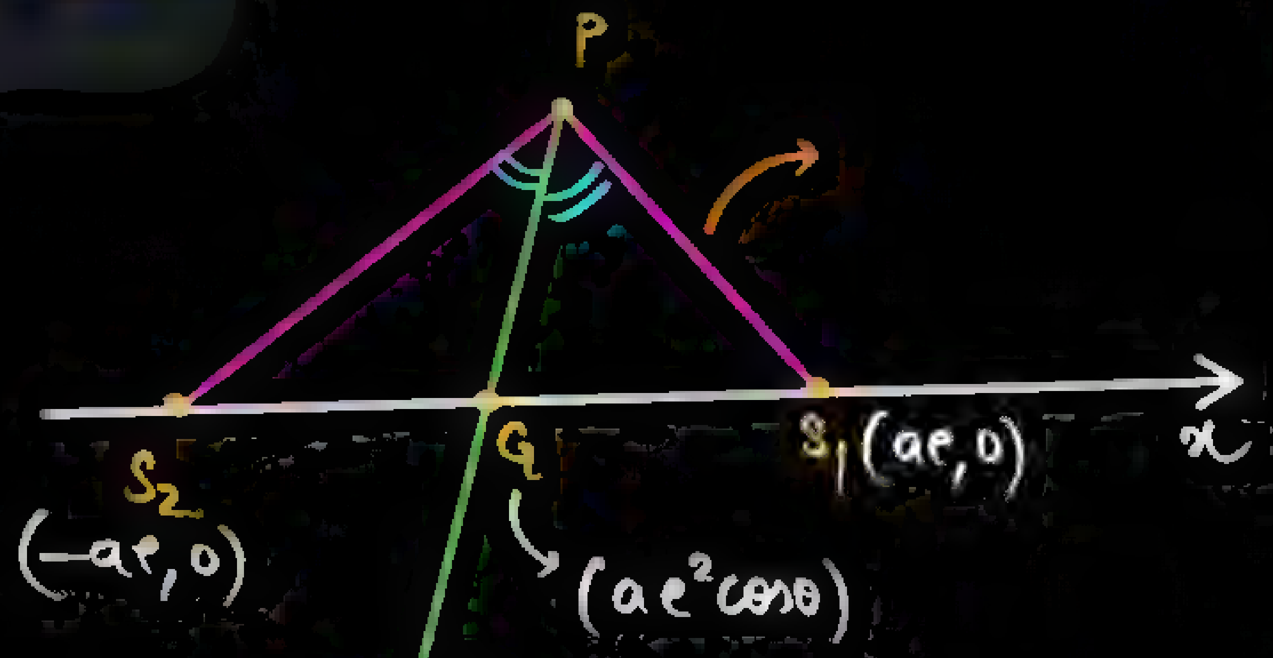
P-5: Tangent and Normal at any point P bisects the angle between focal distances (PS_1 & PS_2).

(ii)

Image of focus in any tangent lies on line joining point of contact & other focus.



Proof:



angle bisector of $\angle P$

eq^n: $\boxed{\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2e^2}$

G $(y=0)$

$$\frac{ax}{\cos\theta} = a^2e^2 \quad (x = ae^2\cos\theta)$$

To prove: $\boxed{\dfrac{S_2G}{S_1G} = \dfrac{PS_2}{PS_1}}$

$$LHS = \frac{S_2G}{S_1G} = \frac{ae^2\cos\theta + ae}{ae - ae^2\cos\theta} = \frac{\cancel{ae}(e\cos\theta + 1)}{\cancel{ae}(1 - e\cos\theta)}$$

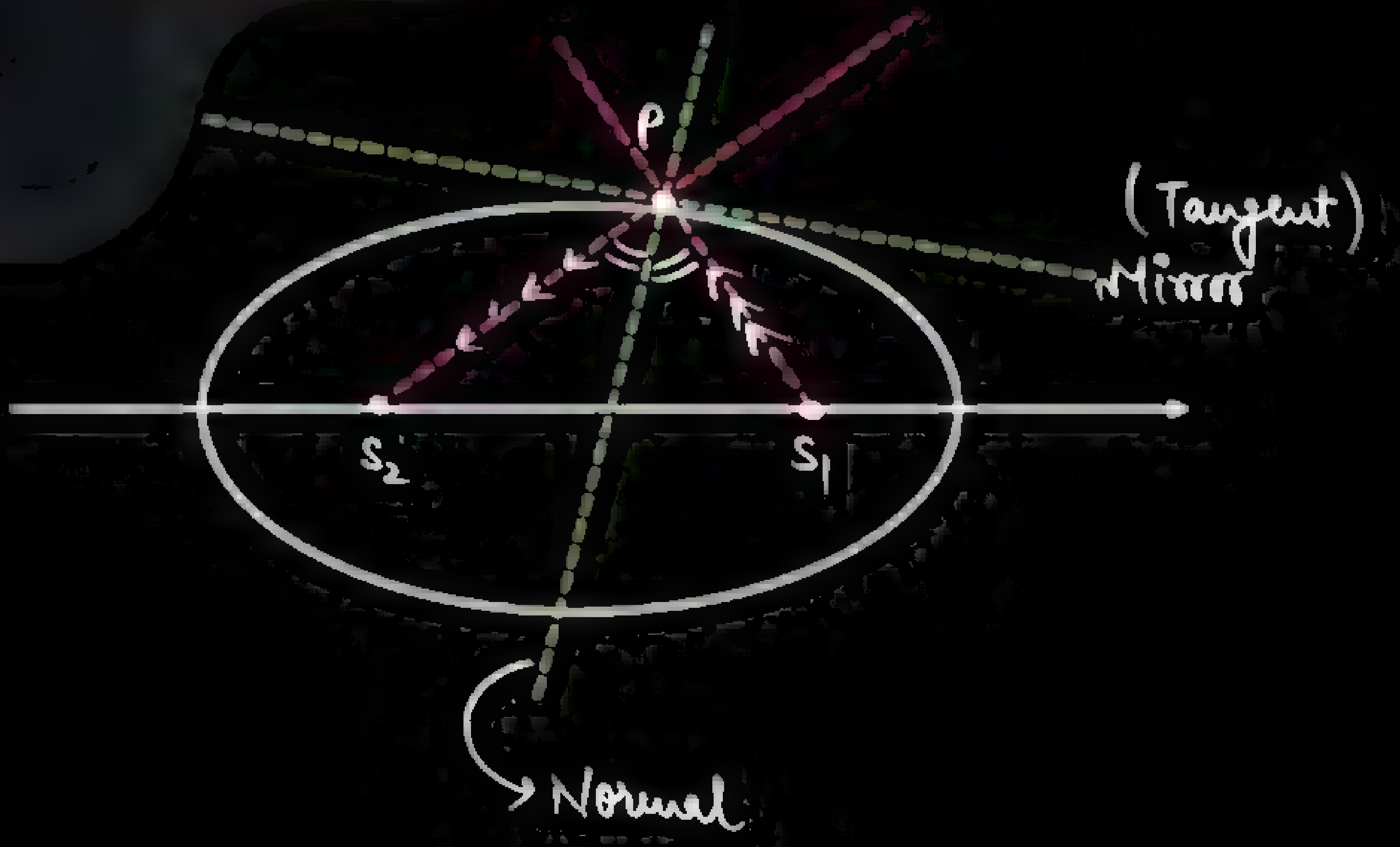
$$\Rightarrow RHS = \frac{PS_2}{PS_1} = \frac{\cancel{e}\left(\frac{a}{e} + a\cos\theta\right)}{\cancel{e}\left(\frac{a}{e} - a\cos\theta\right)} = \left(\frac{1 + e\cos\theta}{1 - e\cos\theta}\right)$$

Same

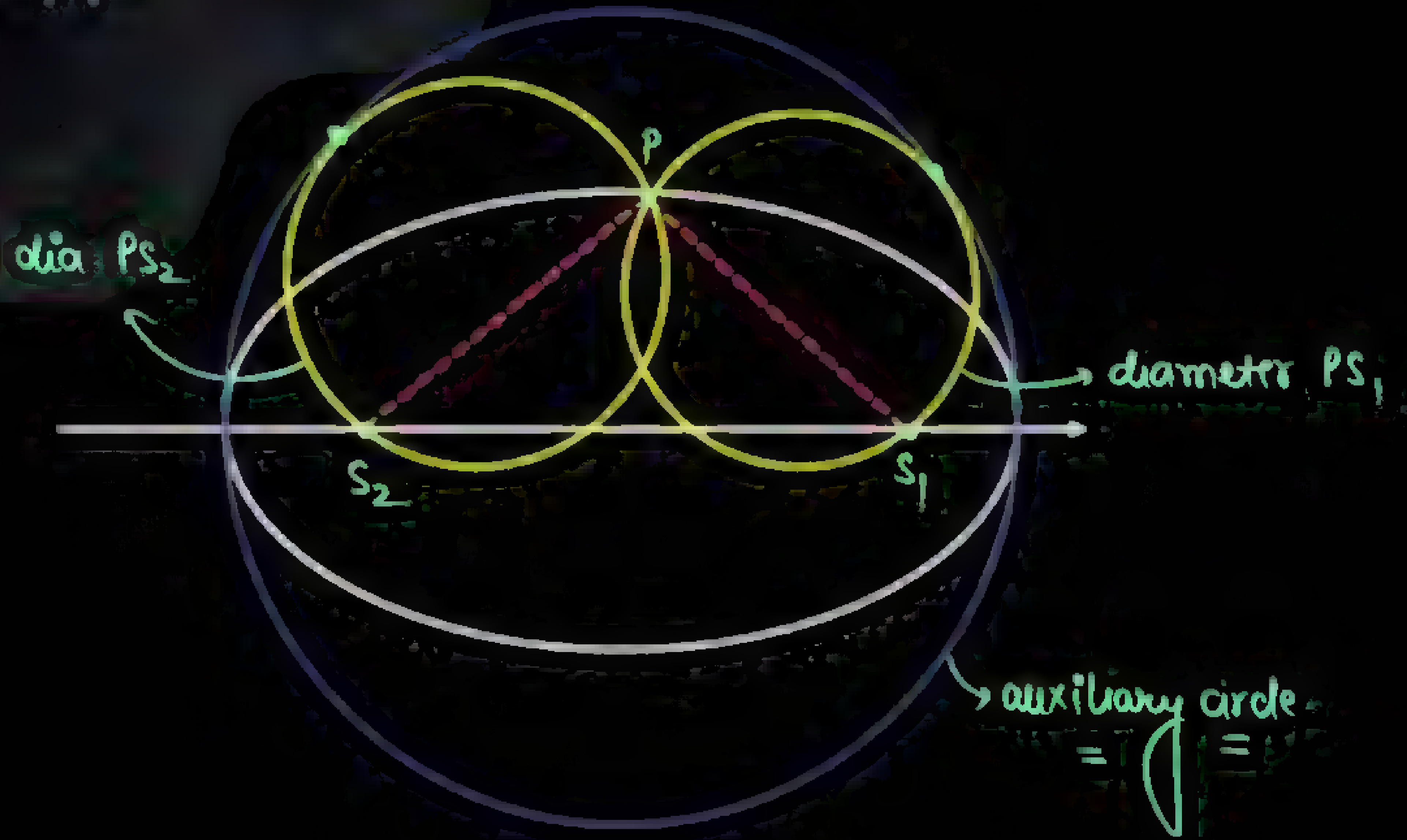
HHPP

REFLECTION PROPERTY:

Any ray passing through one focus, after reflection from Ellipse passes from another focus.



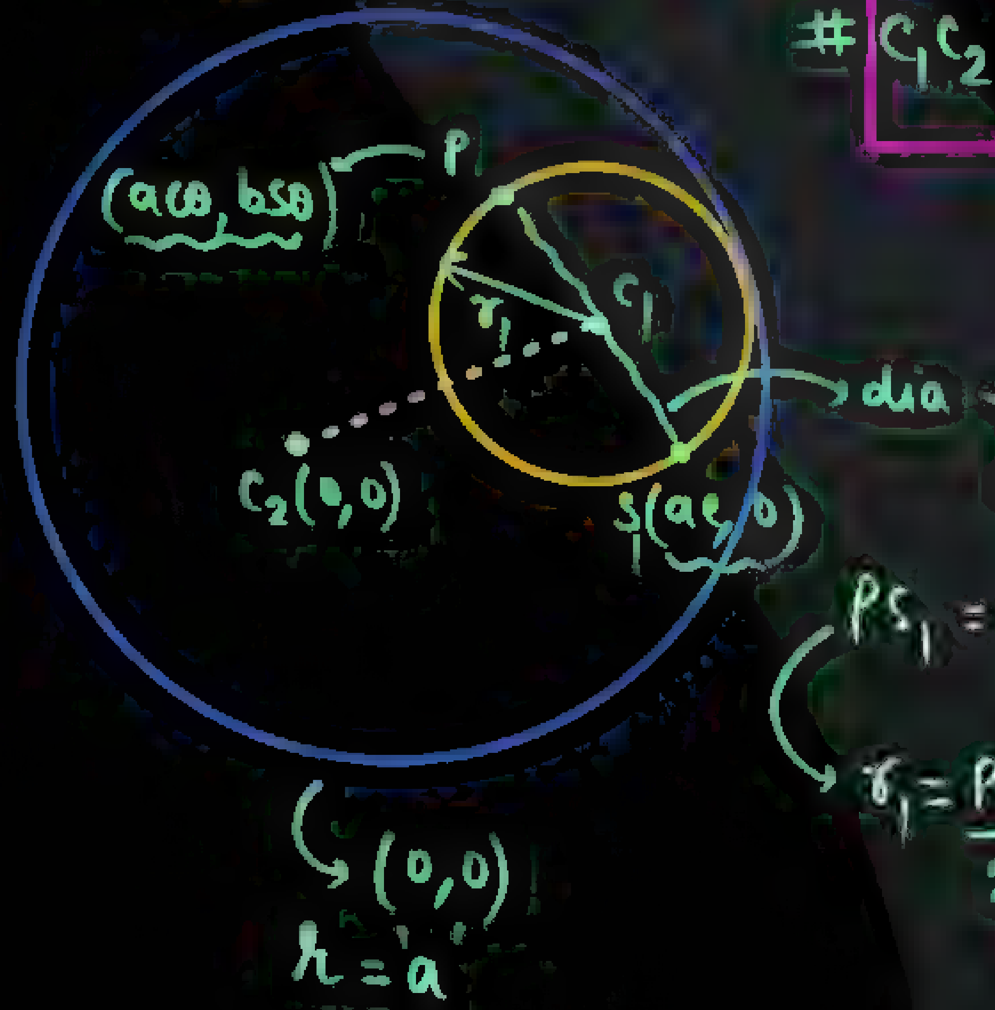
P-6: Circle with focal distance as diameter touches auxiliary circle.



Proof :

To prove :

$\boxed{C_1C_2 = a - r_1}$



$PS_1 = a - ae\cos\theta$

$r_1 = \frac{PS_1}{2}$

Note :

$PA \cdot PB = PT^2$

= (Power of point P)

= (point P ke coord. circle ki eq^n mein put krdo.)

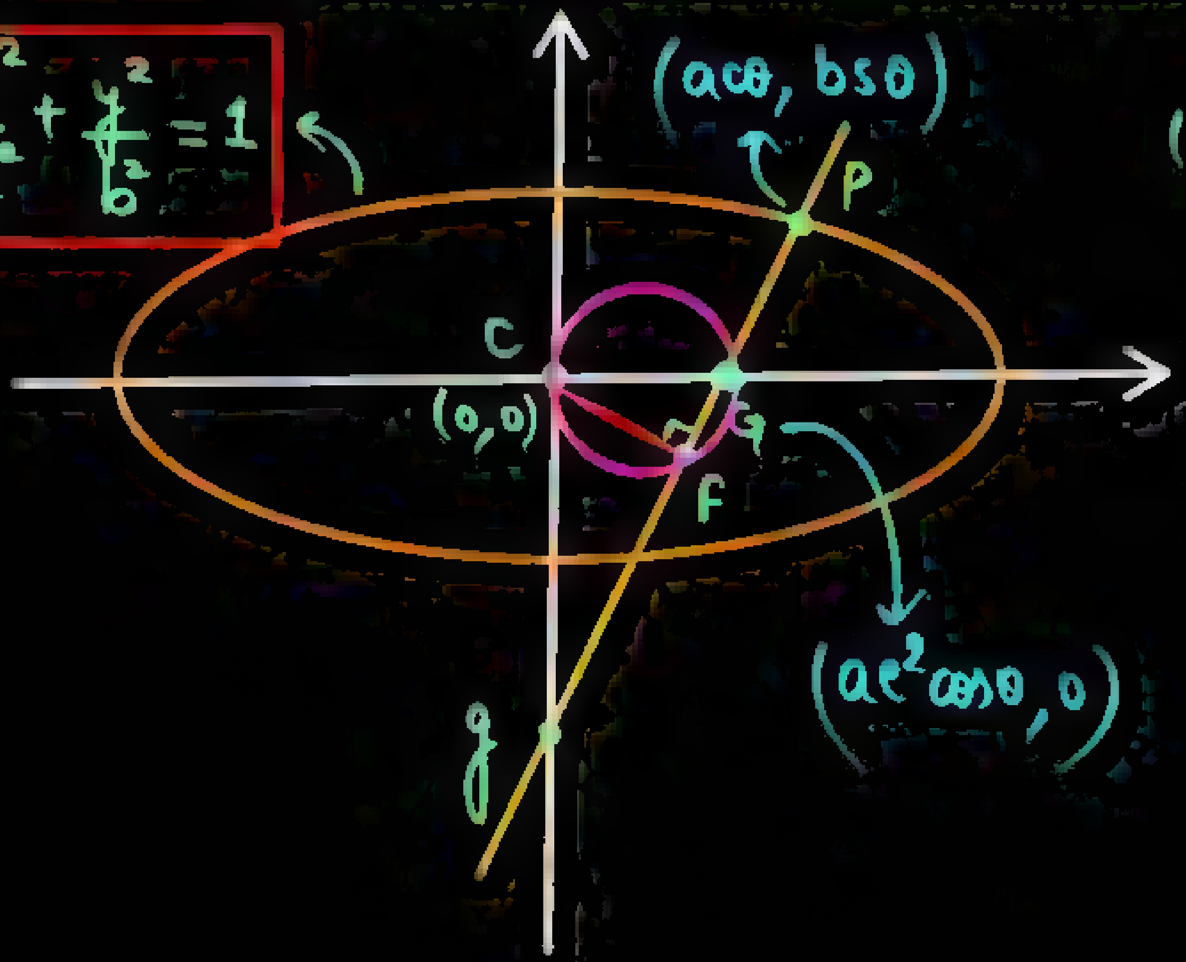
P-7: *If the normal at any point P on the ellipse with centre C meet the major and minor axes in G & g respectively, and if CF be perpendicular upon this normal, then*

(i) $PF \cdot PG = b^2$ (ii) $PF \cdot Pg = a^2$

(iii) $PG \cdot Pg = SP \cdot S'P$ (iv) $CG \cdot CT = CS^2$

(where T is the point where Tangent at P cuts major axis)

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$



(i) Proof : Circle with C & G as dia :-

$$x(x - ae^2\cos\theta) + y^2 = 0$$

$$\boxed{x^2 + y^2 - (ae^2\cos\theta)x = 0}$$

$PF \cdot PG = (\text{power of point } P) = S_1$

$= (ac\theta)^2 + (bs\theta)^2 - ae^2c\theta \cdot ac\theta$

$a^2c^2\theta + b^2s^2\theta - a^2e^2c^2\theta$

$a^2c^2\theta(1 - e^2) + b^2s^2\theta$

$\cancel{a^2}c^2\theta \dfrac{b^2}{\cancel{a^2}} + b^2s^2\theta$

\# $b^2 = b^2(c^2\theta + s^2\theta)$

P-7: *If the normal at any point P on the ellipse with centre C meet the major and minor axes in G & g respectively, and if CF be perpendicular upon this normal, then*

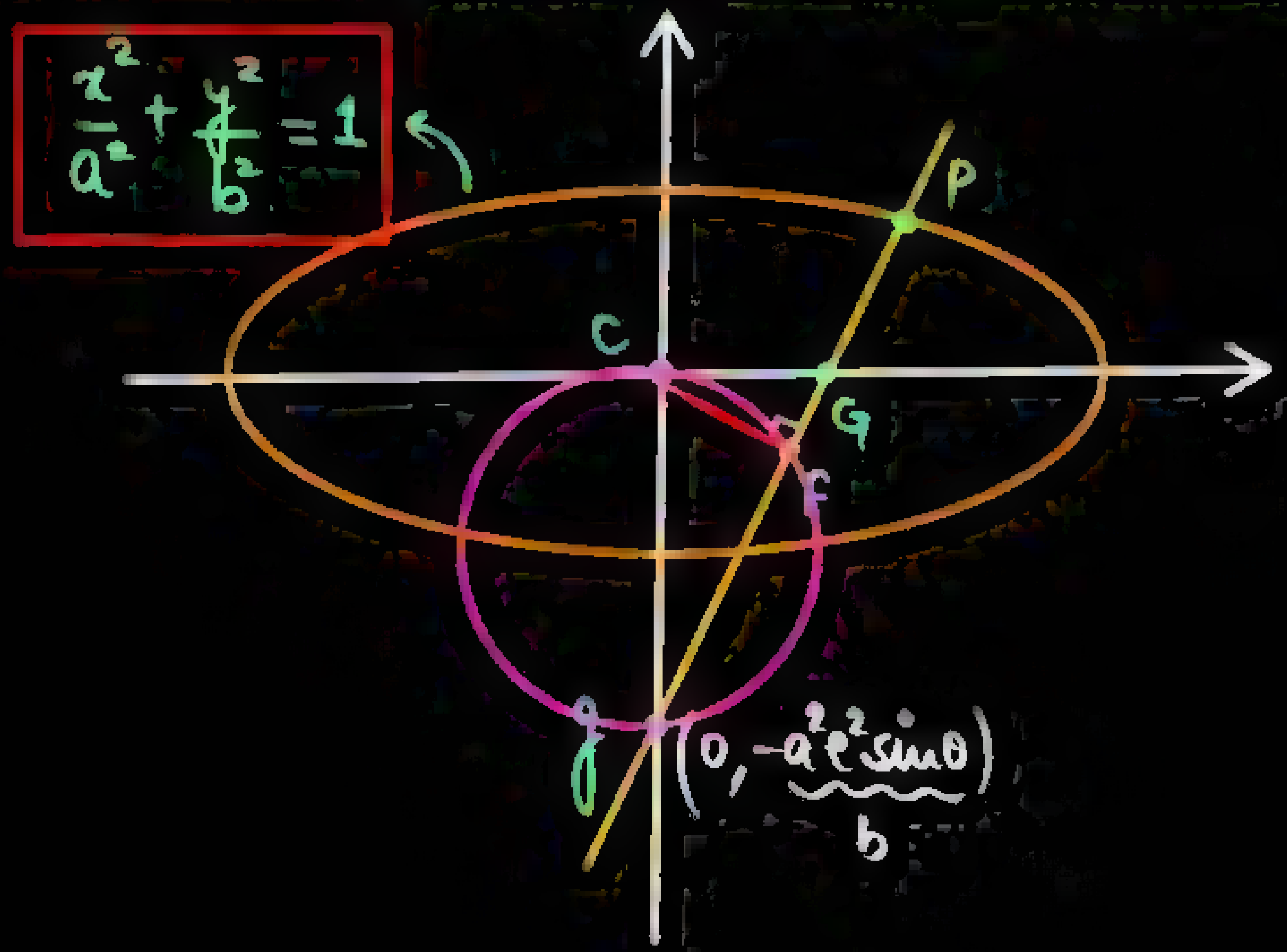
(i) $PF \cdot PG = b^2$

~~(ii)~~ $PF \cdot Pg = a^2$

(iii) $PG \cdot Pg = SP \cdot S'P$

(iv) $CG \cdot CT = CS^2$

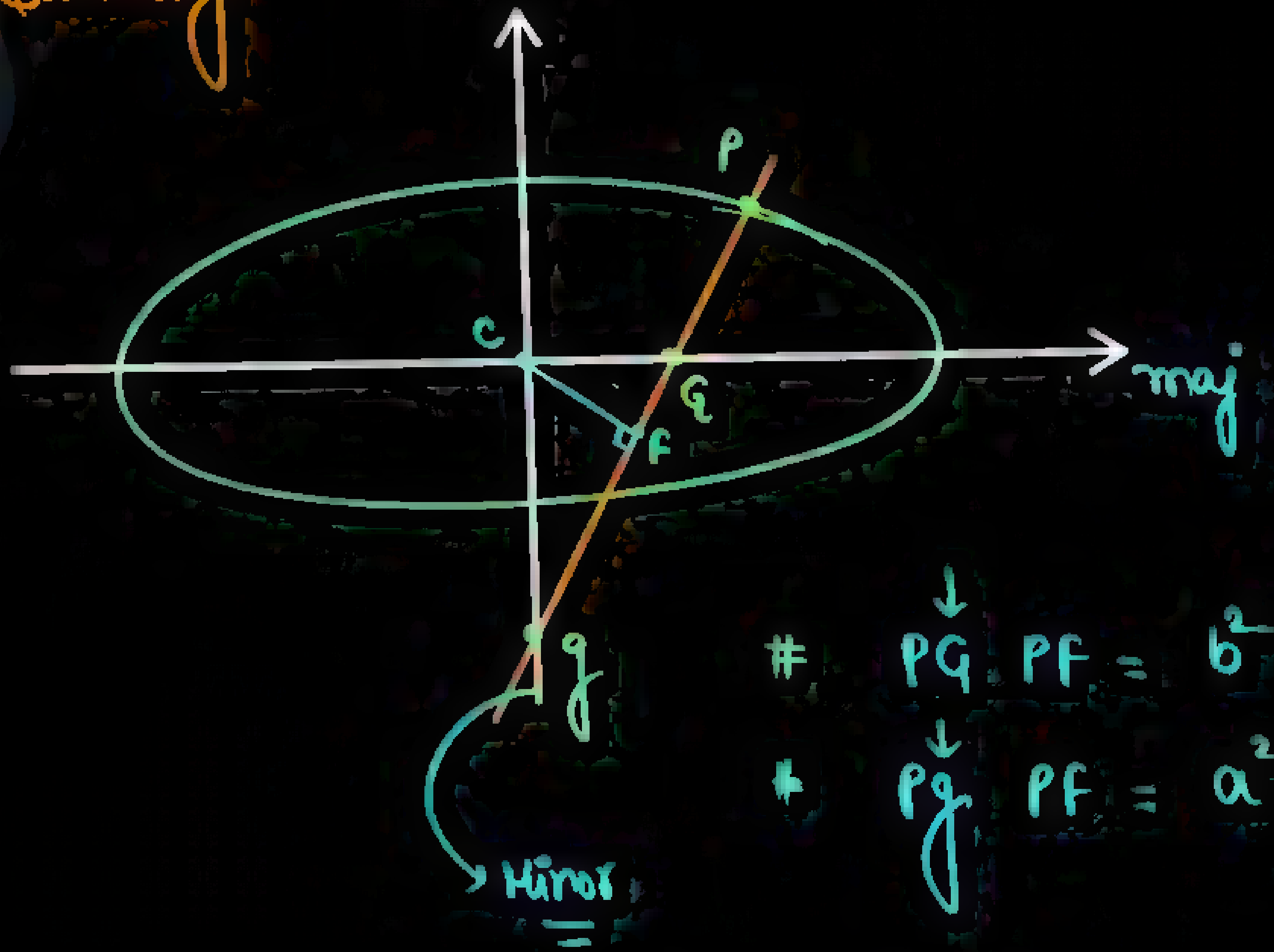
(where T is the point where Tangent at P cuts major axis)



(ii) Circle with C & g as diam :-

Power of point P = PF · Pg

Summary



$PG \cdot PF = b^2$

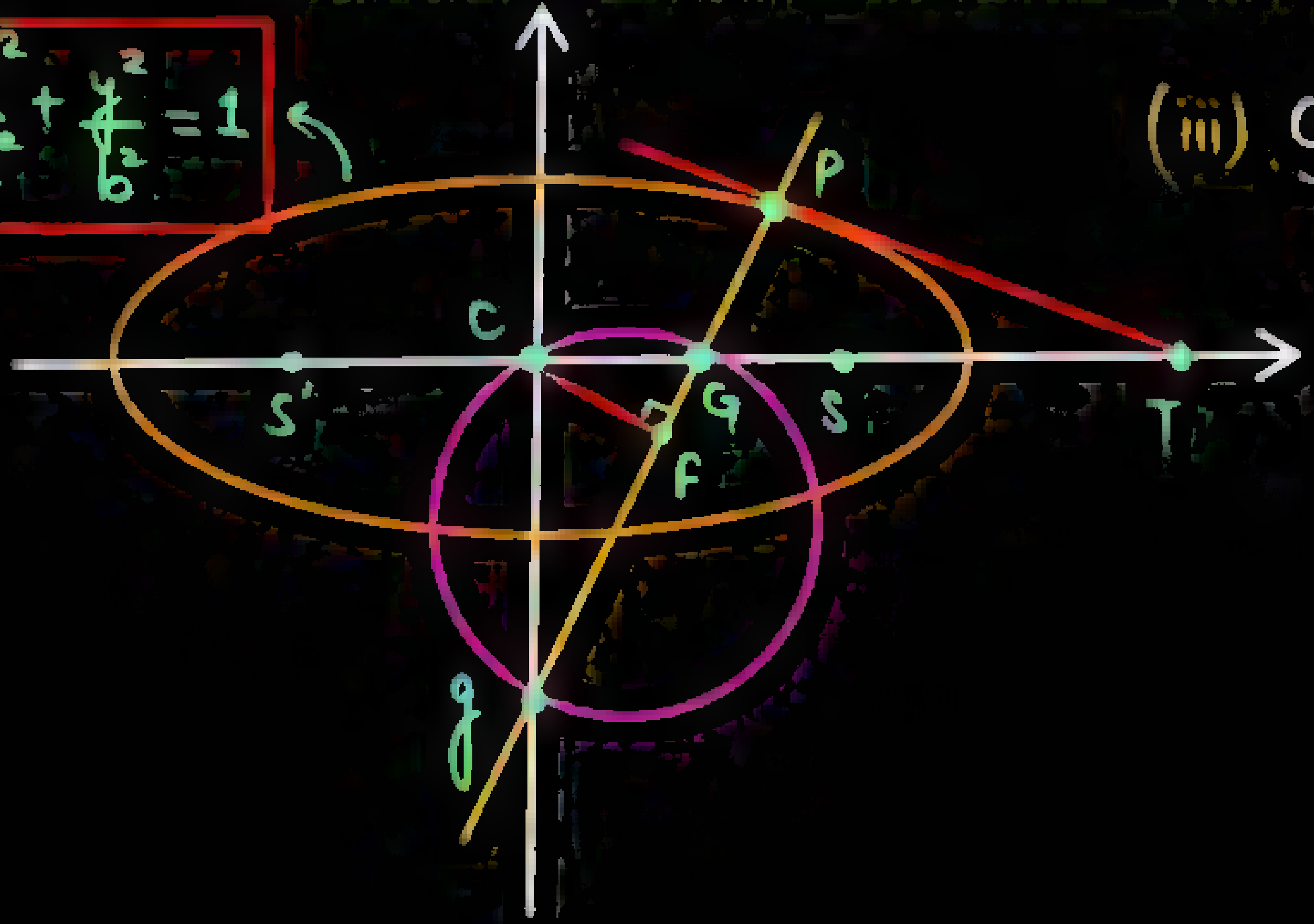
$Pg \cdot PF = a^2$

P-7: If the normal at any point P on the ellipse with centre C meet the major and minor axes in G & g respectively, and if CF be perpendicular upon this normal, then

(i) $PF \cdot PG = b^2$	(ii) $PF \cdot Pg = a^2$
(iii) $PG \cdot Pg = SP \cdot S'P$	(iv) $CG \cdot CT = CS^2$

(where T is the point where Tangent at P cuts major axis)

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$



(iii) Circle with G & g as diam :

Power of point P = $\boxed{PG \cdot Pg = \text{product of focal distances}}$

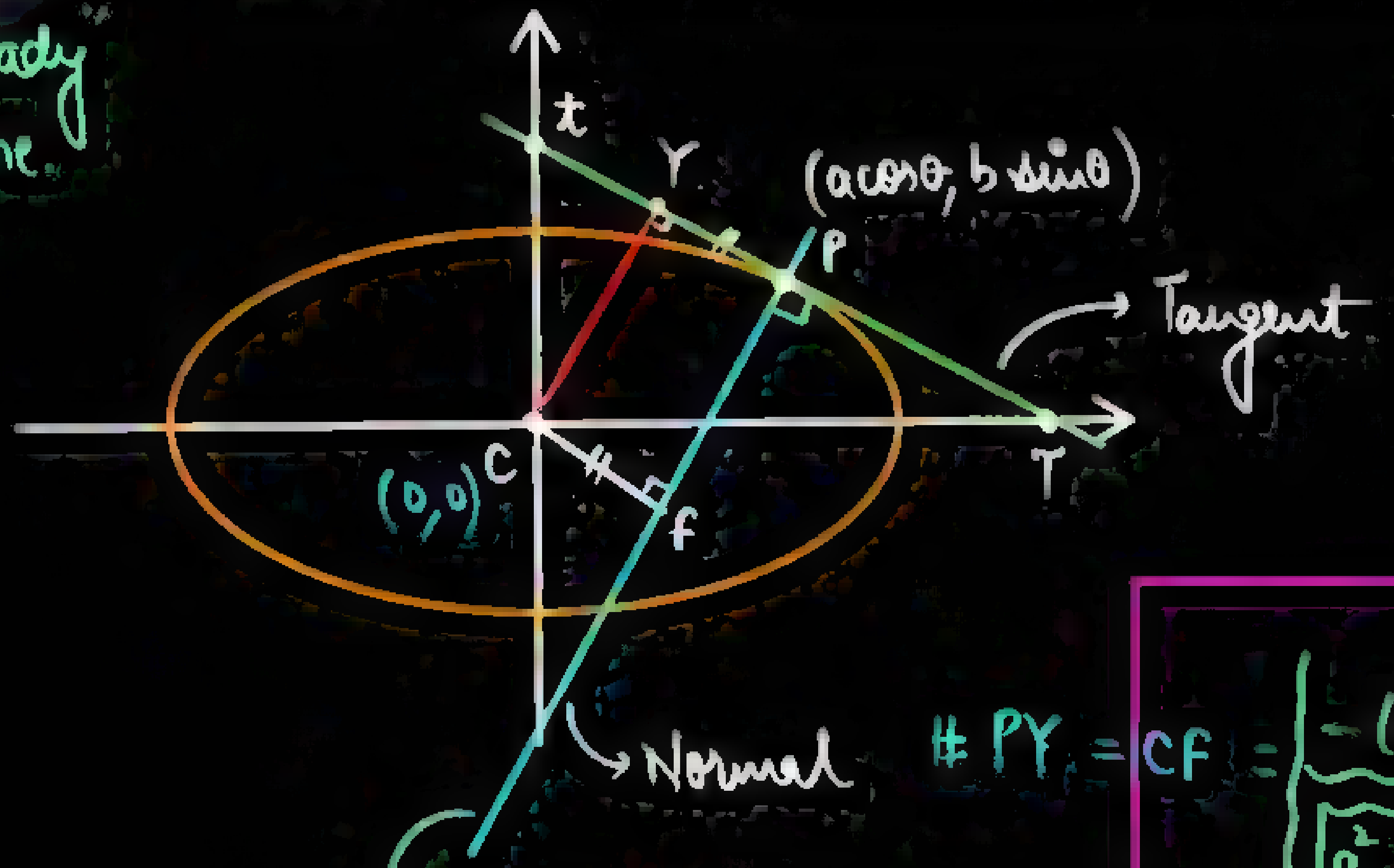
P-8: If tangent at point P meets axes of standard ellipse at T & t and CY is perpendicular on it from centre then :

(i) (Tt) (PY) = $a^2 - b^2$

(ii) Least value of (Tt) = (a + b)

*** (iii) Maximum distance of normal from centre = (a – b)

already done



$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} - (a^2-b^2) = 0$

$\# \; T_P : \boxed{\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1}$

$T \equiv (a\sec\theta, 0)$

$t \equiv (0, b\,\mathrm{cosec}\,\theta)$

$\# \; Tt = \sqrt{a^2\sec^2\theta + b^2\mathrm{cosec}^2\theta}\Big|_{min} = (a+b)$

$\# \; PY = CF = \boxed{\left|\frac{-(a^2-b^2)}{\sqrt{\frac{a^2}{\cos^2\theta}+\frac{b^2}{\sin^2\theta}}}\right|} = \frac{(a^2-b^2)}{\sqrt{a^2\sec^2\theta+b^2\mathrm{cosec}^2\theta}}$

$\# \; CF \cdot Tt = a^2 - b^2$

$CF\Big|_{max} = \left|\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\mathrm{cosec}^2\theta}}\right|_{max} = \frac{a^2-b^2}{a+b} = (a-b)$

min

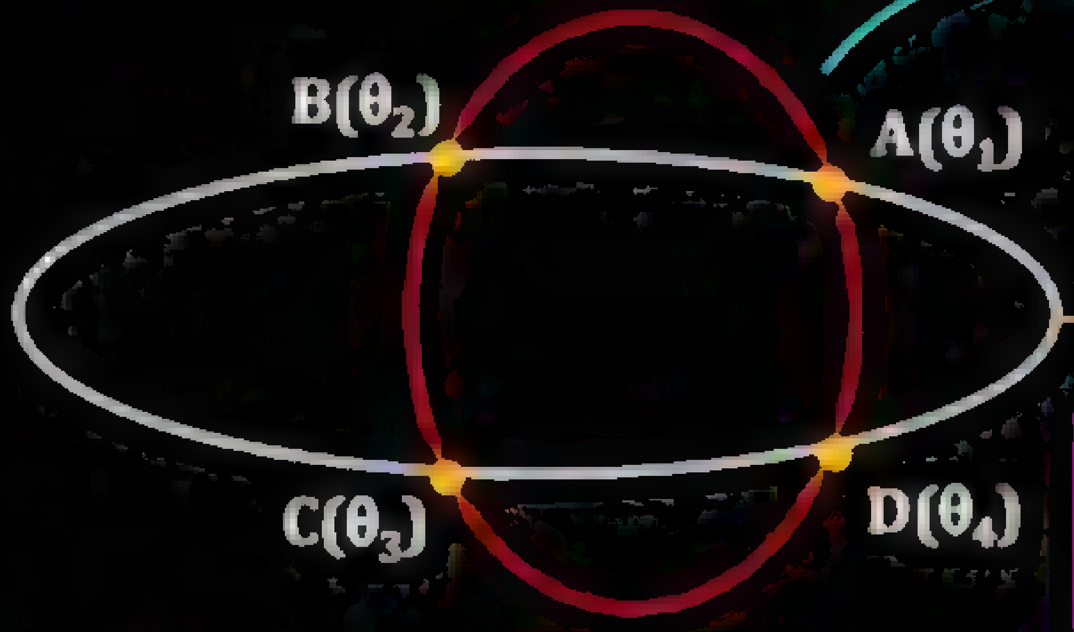
P-09 :

$$\boxed{\begin{aligned}\text{Area of ellipse} &= \pi ab \\ &= \pi(\text{Semi-Major})(\text{Semi-Minor})\end{aligned}}$$

Note:

If any general circle intersect standard ellipse at 4 points (say : $A(\theta_1), B(\theta_2), C(\theta_3)$ & $D(\theta_4)$)

then $\boxed{\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2n\pi}$



Circle : $\boxed{x^2 + y^2 + 2gx + 2fy + c = 0}$

$\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$

solve

Put : $x = a\cos\theta$
$y = b\sin\theta$

$(\cos\theta$ & $\sin\theta)$ → $\tan\dfrac{\theta}{2}$

form a Biquad

$\left(\tan\dfrac{\theta_1}{2}, \tan\dfrac{\theta_2}{2}, \tan\dfrac{\theta_3}{2}, \tan\dfrac{\theta_4}{2}\right)$

$\boxed{S_1 = S_3 = 0}$

$\tan\left(\dfrac{\theta_1}{2} + \dfrac{\theta_2}{2} + \dfrac{\theta_3}{2} + \dfrac{\theta_4}{2}\right) = \dfrac{S_1 - S_3}{1 - S_2 + S_4} = 0$

Q. An ellipse is with major axis = 2a, minor axis = 2b is sliding between coordinate axes, then find locus of centre & focii of ellipse

CHALLENGER

H.W.

HYPERBOLA

BASICS OF HYPERBOLA

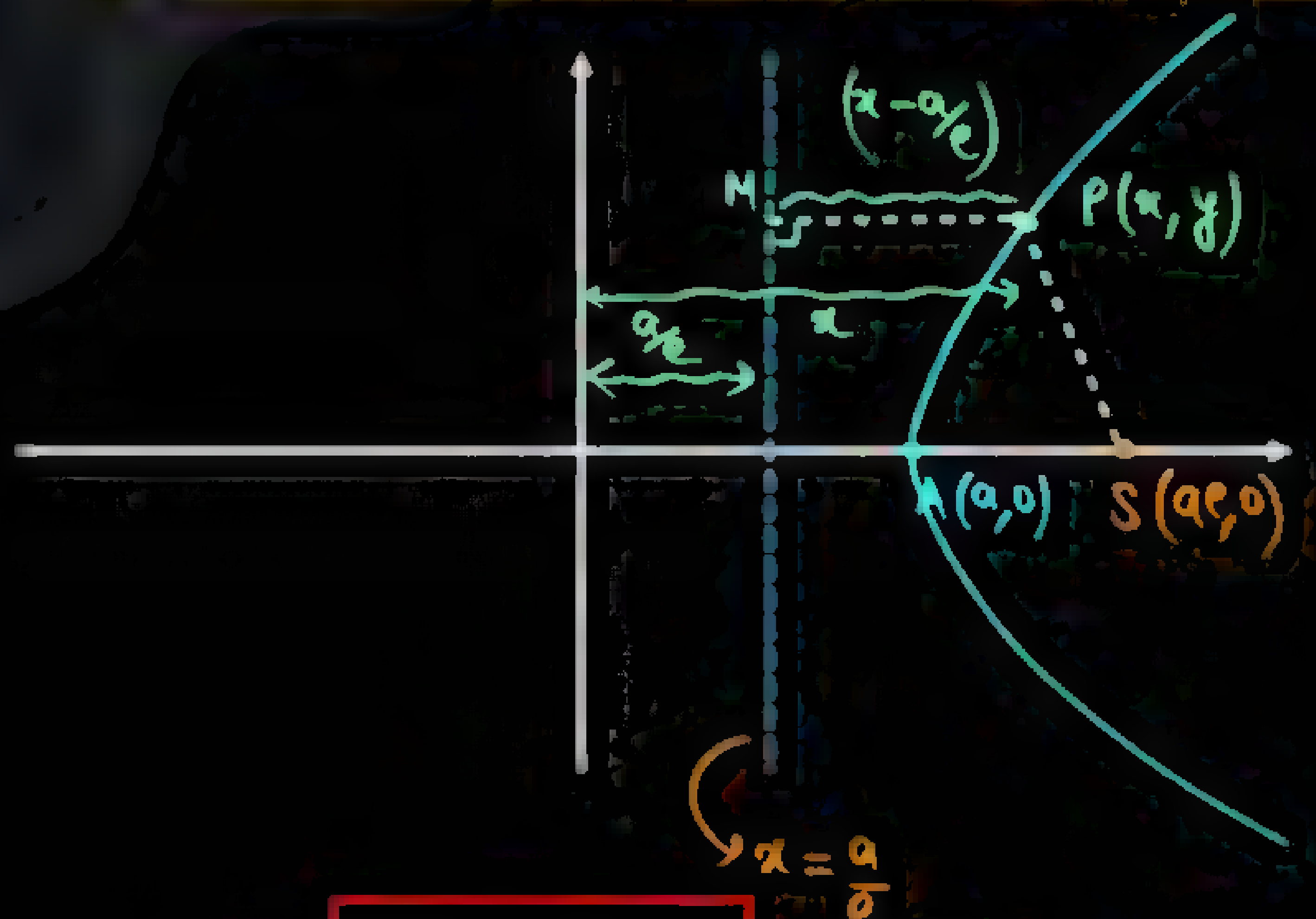
Definition: For Hyperbola $\boxed{e > 1}$

For second degree Equation: $\boxed{ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0}$

Condition: $\boxed{\Delta \neq 0,\ h^2 > ab}$

STANDARD HYPERBOLA

Focus on X-axis & Directrix parallel to Y-axis.



$PS = e\,PM$

$$\boxed{PS^2 = e^2\,PM^2}$$

$$(x-ae)^2 + y^2 = e^2\left(x-\frac{a}{e}\right)^2$$

$$e^2\left(x^2+\frac{a^2}{e^2}-\frac{2ax}{e}\right)$$

$$y^2 + x^2 + a^2e^2 - \cancel{2ae}x = e^2x^2 + a^2 - \cancel{2ae}x$$

$$x^2 - e^2x^2 + y^2 = a^2 - a^2e^2$$

$$(1-e^2)x^2 + y^2 = a^2(1-e^2)$$

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

b^2

$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$

$b^2 = a^2(e^2-1)$

Very Important BAAT :

Ellipse

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \xrightarrow{b^2 \to (-b^2)} \boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$

Hyperbola

"Many results" of ellipse can be converted into results for Hyperbola by replacing b^2 by $(-b^2)$

Draw :

$\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$

Symm. w.r.to X-axis & Y-axis

P.O.I with x-axis

Put : $\boxed{y = 0}$

$\dfrac{x^2}{a^2} = 1 \Rightarrow x = \pm a$

P.O.I. with y-axis

$\boxed{x = 0}$

$-\dfrac{y^2}{b^2} = 1 \Rightarrow \boxed{y^2 = -b^2} \Rightarrow \phi$

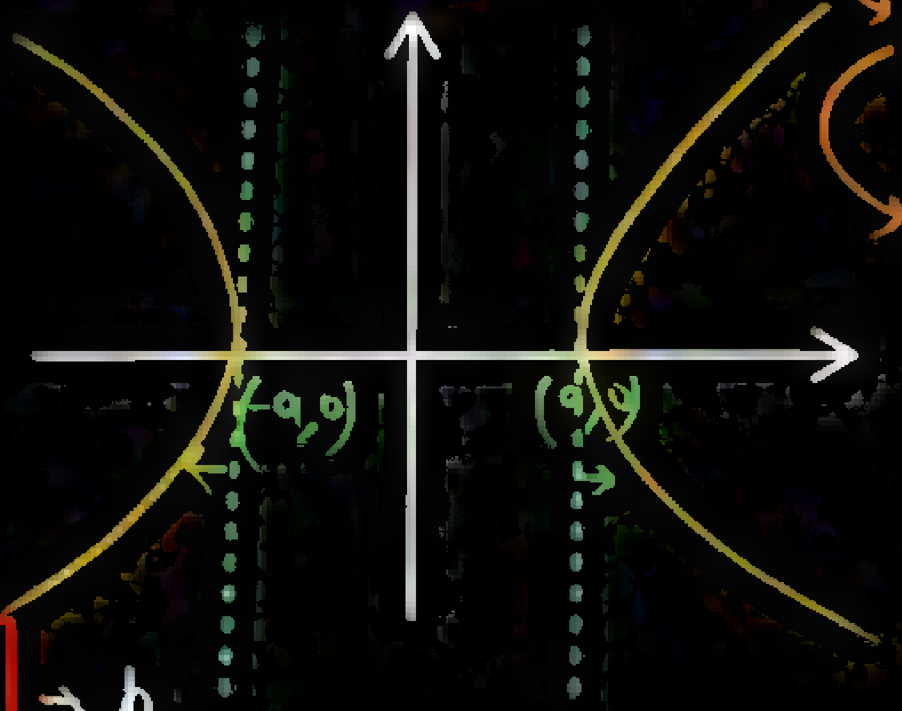
$\dfrac{x^2}{a^2} - 1 = \dfrac{y^2}{b^2}$

RHS ≥ 0

LHS $= \dfrac{x^2}{a^2} - 1 \geq 0 \Rightarrow x^2 - a^2 \geq 0$

$(x-a)(x+a) \geq 0$

$x \in (-\infty, -a] \cup [a, \infty)$



$\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ → intersects x-axis but not y-axis

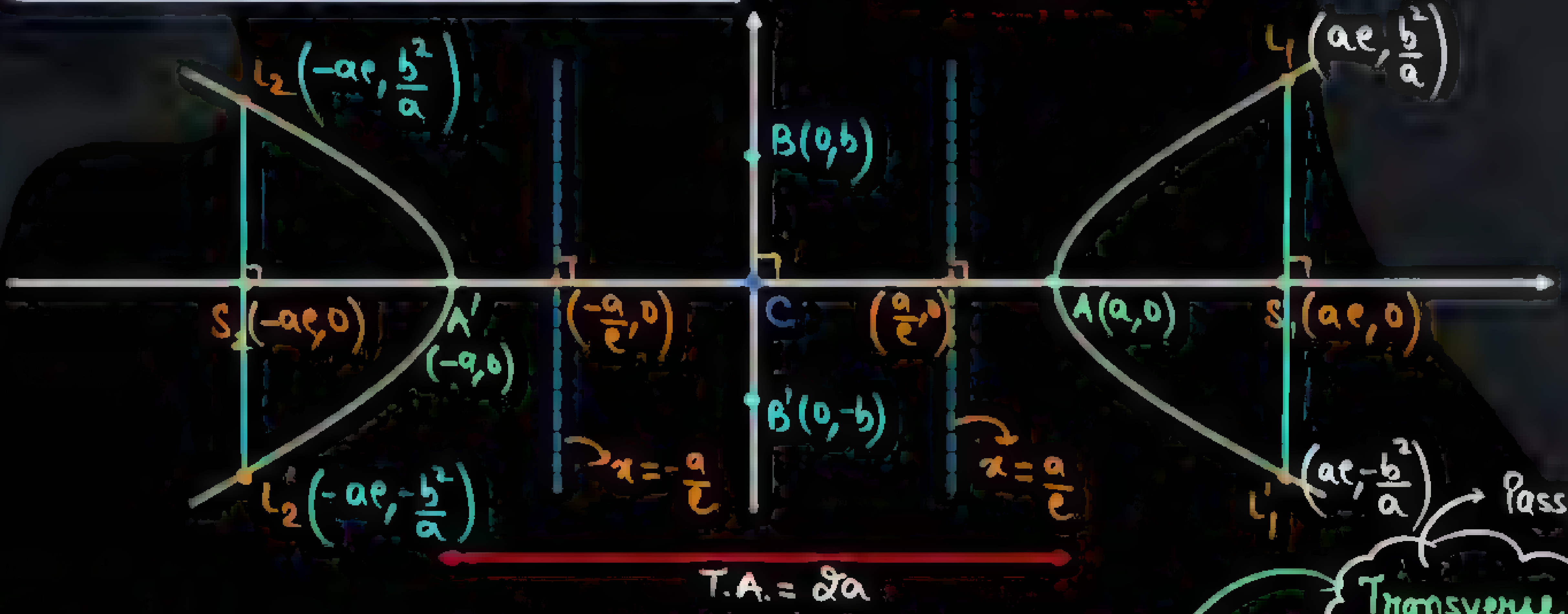
$\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1}$ → intersects y-axis but not x-axis

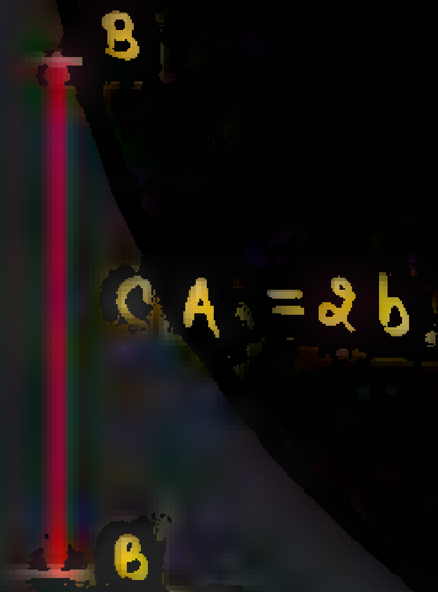
"Bada chota"

→ HB ×

COMPLETE HYPERBOLA

$\boxed{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1}$





Passing thru S_1 & S_2

Transverse axis → $AA = 2a$

Conjugate Axis → $\perp$ to T.A. & passing thru centre → $BB = 2b$

Foci: S_1 & $S_2 \equiv (\pm ae, 0)$

Directrices: $x = \pm\frac{a}{e}$

Vertices: A' & $A \equiv (\pm a, 0)$

Axes / Principle Axes:

Centre: P.O.I. of T.A. & C.A. $(C \equiv (0,0))$

Focal Length: $S_1S_2 = 2ae$

ECCENTRICITY & LATUS RECTUM

$b^2 = a^2(e^2-1)$

$$\frac{b^2}{a^2} = e^2 - 1$$

$\boxed{1 + \frac{b^2}{a^2} = e^2}$

$$e^2 = 1 + \left(\frac{2b}{2a}\right)^2$$

$$\boxed{e^2 = 1 + \left(\frac{C.A}{T.A}\right)^2}$$ ***

$LR = \frac{2b^2}{a}$

$$\boxed{L.R = \frac{(C.A)^2}{(T.A)}}$$ ***

or

$$\boxed{L.R = 2e(\text{distance b/w focus \& corresponding directrix})}$$

distance b/w S & CD directrix

$= ae - \frac{a}{e}$

Ex. Find 'e' & draw diagram of following :

(i) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

T.A = 6 # C.A = 4

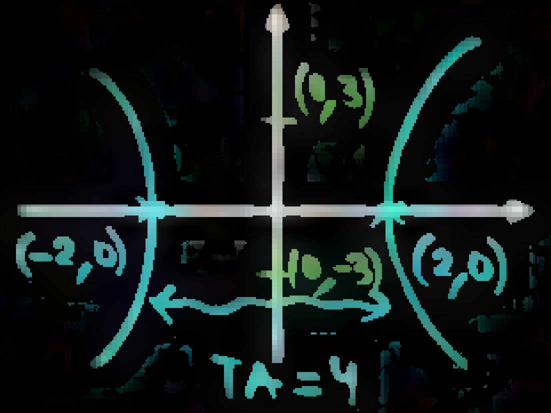
$e^2 = 1 + \left(\frac{4}{6}\right)^2 = 1 + \frac{4}{9}$

$\ell R = \frac{(4)^2}{6} = \frac{16}{6} = \frac{8}{3}$

$e^2 = \frac{13}{9} \Rightarrow e = \frac{\sqrt{13}}{3}$

conj.

(ii) $\frac{x^2}{4} - \frac{y^2}{9} = 1$



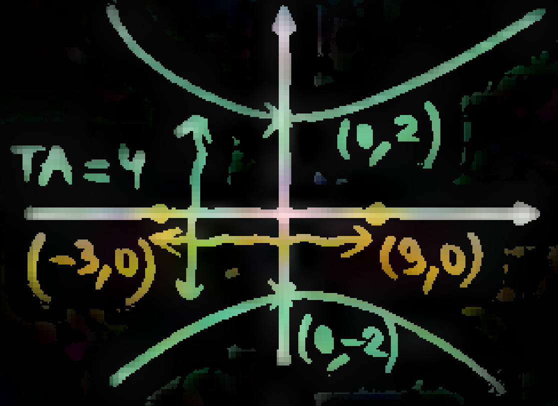
TA = 4, CA = 6

$e^2 = 1 + \left(\frac{6}{4}\right)^2 = 1 + \left(\frac{3}{2}\right)^2 = 1 + \frac{9}{4} = \frac{13}{4} \Rightarrow e = \frac{\sqrt{13}}{2}$

$\ell R = \frac{(6)^2}{4} = \frac{36}{4} = 9$

(iii) $\frac{x^2}{9} - \frac{y^2}{4} = -1$

$-\frac{x^2}{9} + \frac{y^2}{4} = 1$



TA = 4, CA = 6

↳ $e = \sqrt{13}/2$, $\ell R = 9$

conj.

(iv) $\frac{x^2}{4} - \frac{y^2}{9} = -1$

$-\frac{x^2}{4} + \frac{y^2}{9} = 1$

TA = 6, CA = 4 $e^2 = 1 + \frac{4}{9} = \frac{13}{9} \Rightarrow \frac{\sqrt{13}}{3} = e$

$\ell R = \frac{2(4)}{3} = \frac{8}{3}$

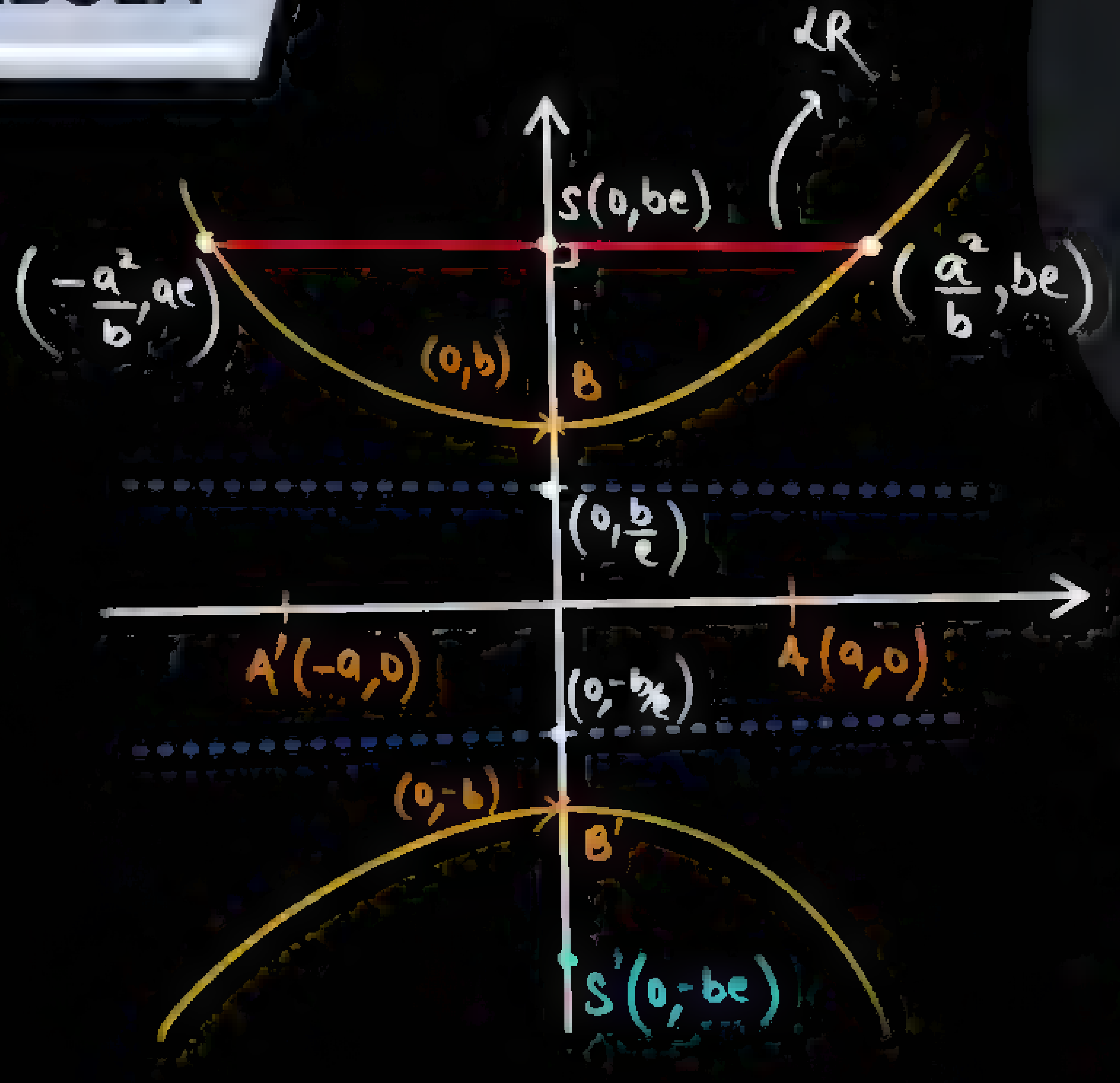
ANOTHER HYPERBOLA

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1}$$

$$\boxed{-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

$$e^2 = 1+\left(\frac{2a}{2b}\right)^2$$

$$\boxed{e^2 = 1+\frac{a^2}{b^2}}$$



Vertices ≡ B & B'

$\rightarrow (0, \pm b)$

Axes:

T.A = BB' = 2b

C.A = AA' = 2a

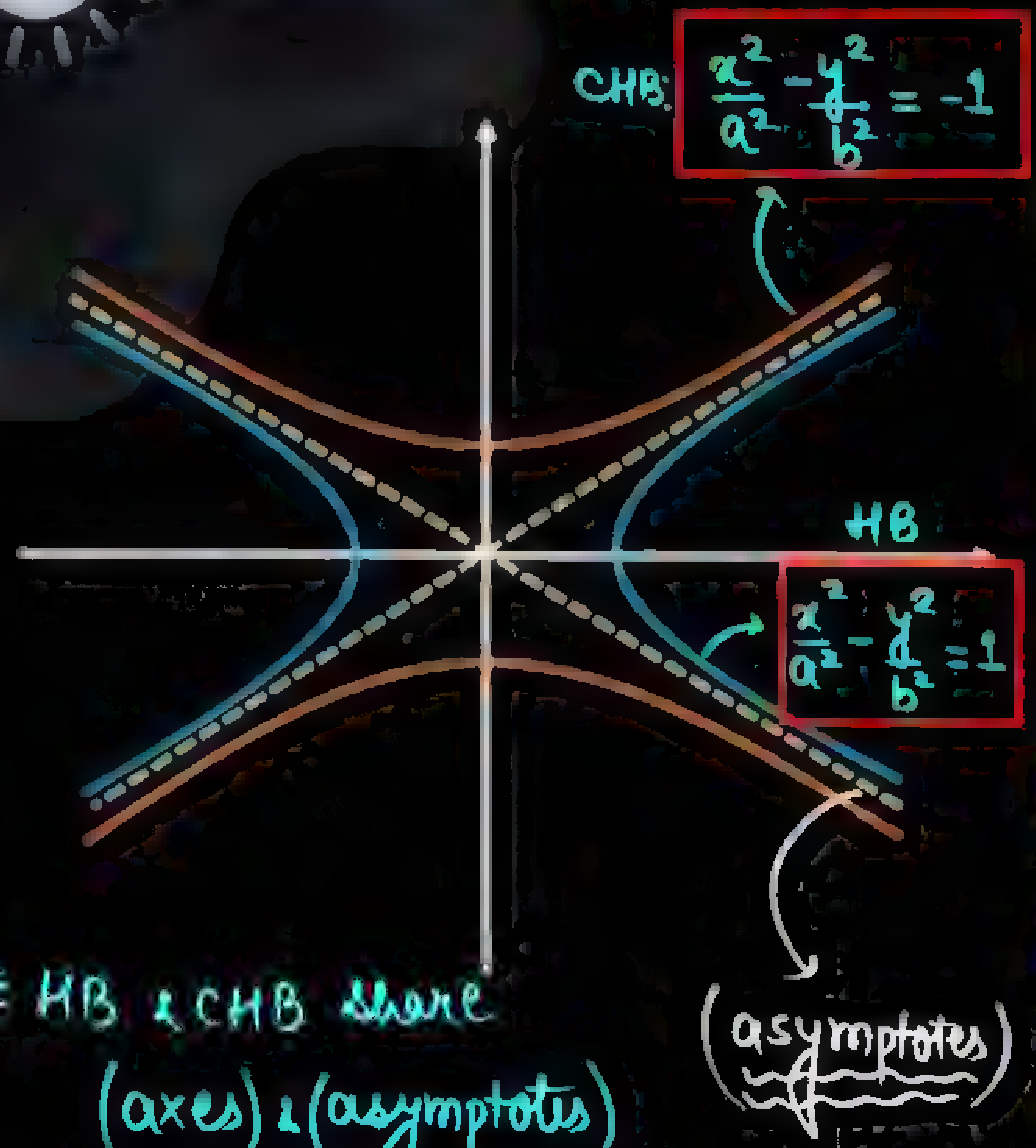
$e^2 = 1+\frac{a^2}{b^2}$

LR $= \frac{2a^2}{b}$

Directrix $\Rightarrow y = \pm\frac{b}{e}$

foci $\equiv (0, \pm be)$

HYPERBOLA & CONJUGATE HYPERBOLA



HB & CHB share (axes) & (asymptotes)

	Hyperbola ($\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$)	Conjugate Hyperbola ($\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$)
Directrices:	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Foci:	$(\pm ae, 0)$	$(0, \pm be)$
Vertices:	$(\pm a, 0)$	$(0, \pm b)$
Principle Axes:	$\boxed{T.A. = 2a, C.A = 2b}$	$\boxed{T.A. = 2b, C.A = 2a}$
Centre:	$(0, 0)$	$(0, 0)$
Focal Length:	$\boxed{2ae}$	$\boxed{2be}$
Eccentricity:	$\boxed{e_1^2 = 1 + \frac{b^2}{a^2}}$	$\boxed{e_2^2 = 1 + \frac{a^2}{b^2}}$
Latus Rectum:	$\boxed{LR = \frac{2b^2}{a}}$	$\boxed{LR = \frac{2a^2}{b}}$

Important Results: (HB & CHB)

Result-01 : The foci of a hyperbola and its conjugate are concyclic and form the vertices of square.

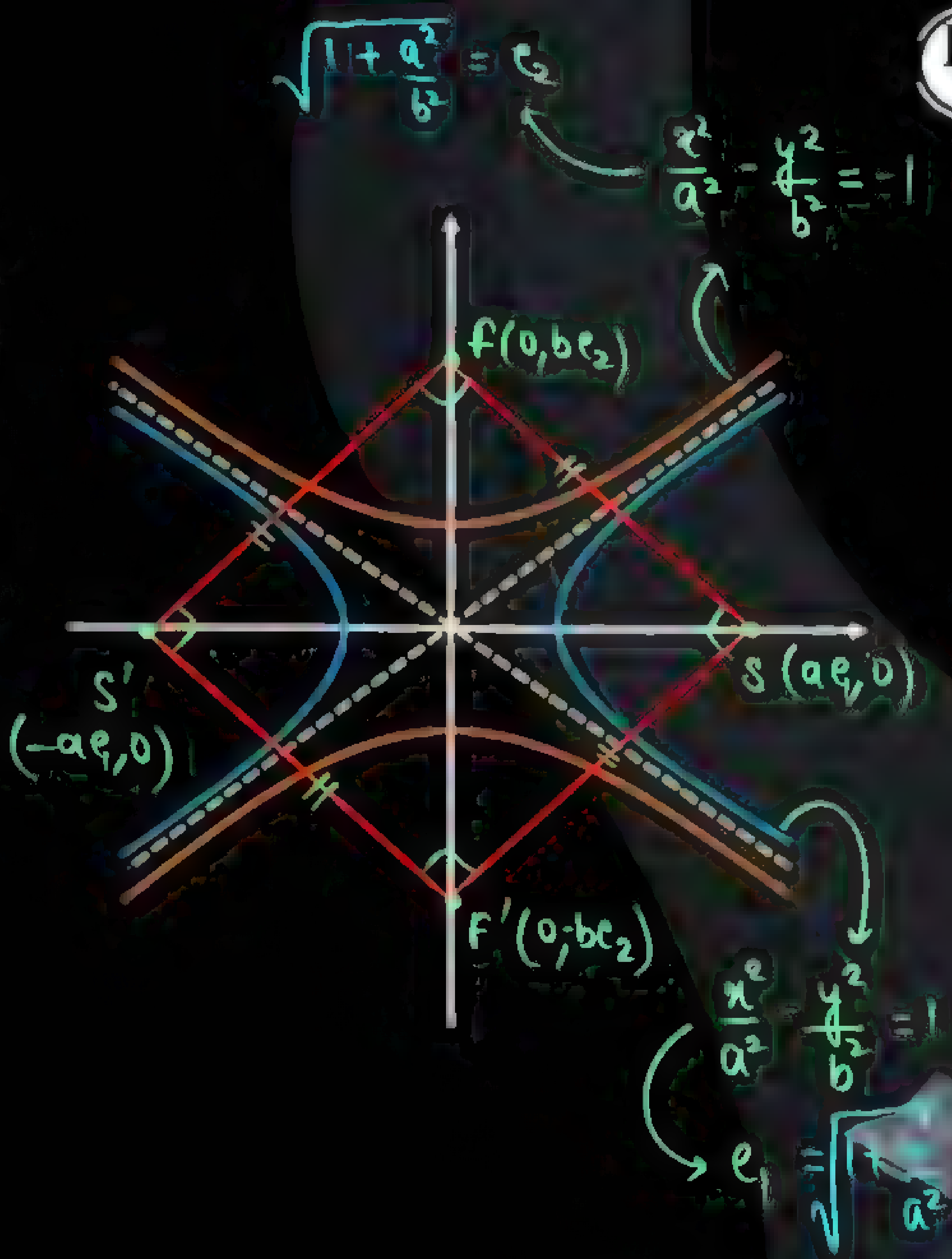
Result-02 : If e_1 & e_2 are eccentricities of hyperbola and its conjugate then :

$$\boxed{\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1}$$

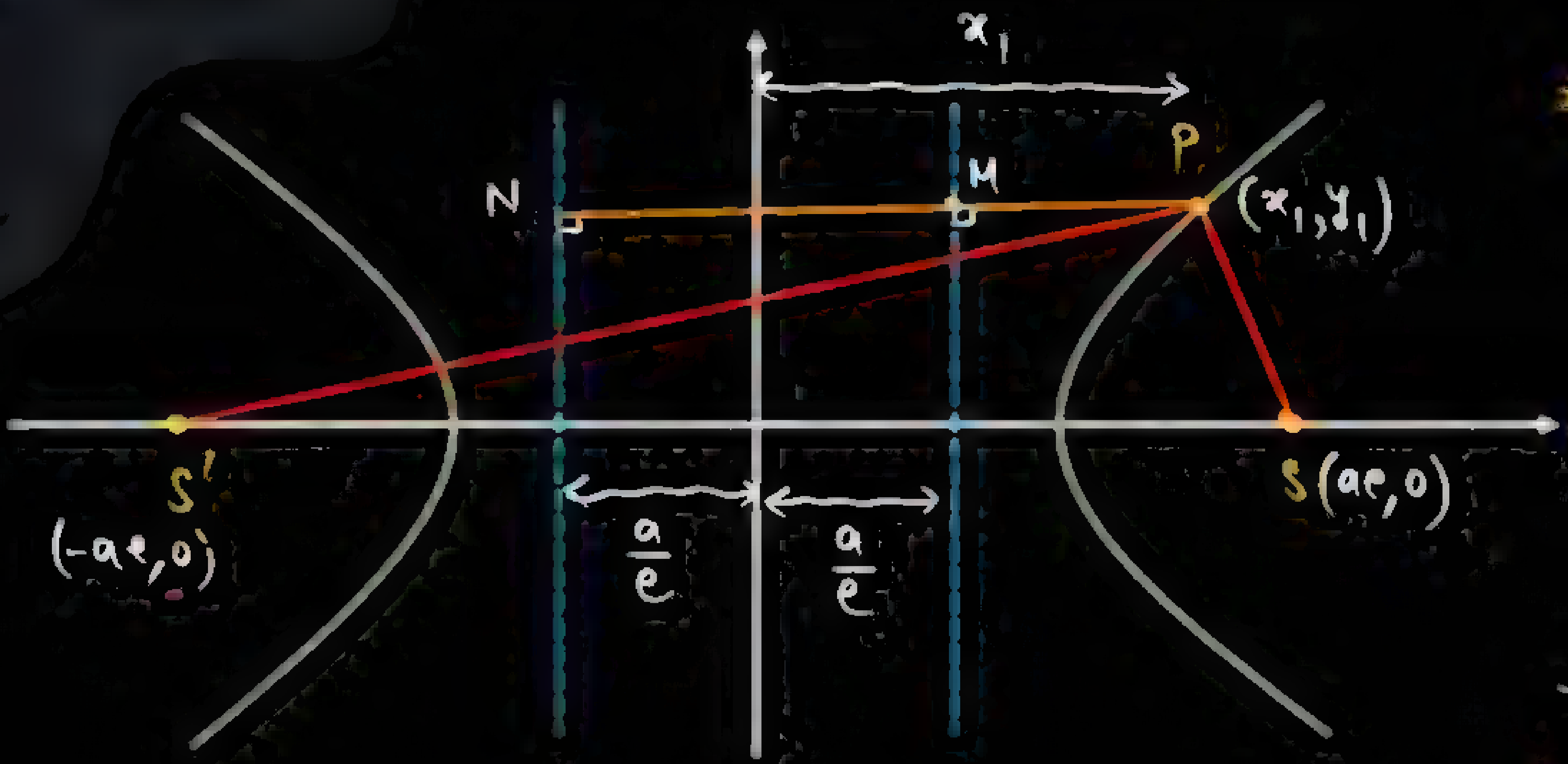
$e_1^2 = \dfrac{a^2+b^2}{a^2}$

$e_2^2 = \dfrac{a^2+b^2}{b^2}$

$$\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}$$



FOCAL DIRECTRIX PROPERTY



\# $PS = e\,PM$

$$= e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$$

\# $PS' = e\,PN$

$$= e\left(x_1 + \frac{a}{e}\right) = ex_1 + a$$

\# $PS' - PS = (ex_1 + a) - (ex_1 - a)$

$$\boxed{PS' - PS = 2a}$$

\# (independent of x_1)

SECOND DEFINITION OF HB

Locus of point which moves such that difference of its distances from two fixed points is constant.

"foci of HB"

"length of Transverse axis"

Ex. Show that locus of centre of a variable circle which touches two fixed non-intersecting circles externally is hyperbola.

and one doesn't lies inside other

$CC_1 = r_1 + r$

$CC_2 = r_2 + r$

$$\boxed{CC_2 - CC_1 = \underbrace{r_2 - r_1}_{\text{const}}}$$

H.B.

C_1 & C_2 ≡ foci

T.A = $(r_2 - r_1)$

Ex. If HB : $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ passes three foci of E : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then find 'e' of both.

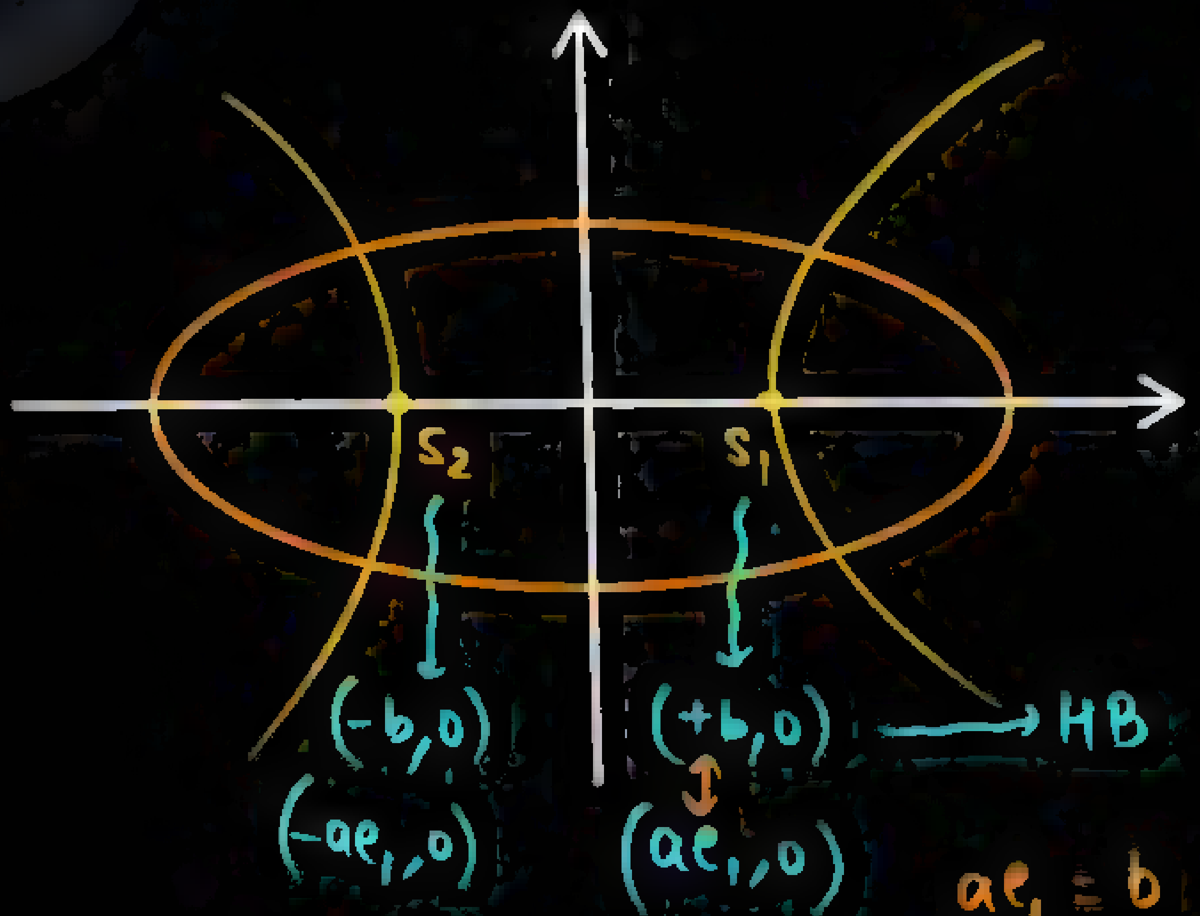
e_2

x-axis intersect

$(\pm b, 0)$

e_1

foci $\equiv S_1$ & S_2



$ae_1 = b$

$e_1 = \frac{b}{a} = \frac{1}{\sqrt{2}}$

$e_2^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \left(\frac{1}{\sqrt{2}}\right)^2$

$= 1 + \frac{1}{2}$

$e_2^2 = \frac{3}{2} \Rightarrow \boxed{e_2 = \sqrt{\frac{3}{2}}}$

$e_1^2 = 1 - \frac{b^2}{a^2}$

$e_1^2 = 1 - e_1^2$

$2e_1^2 = 1$

$e_1^2 = \frac{1}{2}$

$\boxed{e_1 = \frac{1}{\sqrt{2}}}$

Ex. If E : $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ & $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ are confocal then find $b^2=?$

Same foci

H.W.

Ex. Find the equation of the hyperbola whose eccentricity is $\sqrt{2}$ and the distance between the foci is 16, taking transverse and conjugate axes of the hyperbola as x and y-axes respectively.

?

H.W

Ex. Find e of conic whose parametric equation

$$x = \frac{e^t + e^{-t}}{2} \ \& \ y = \frac{e^t - e^{-t}}{3}, t \in R.$$

?

H.W

TODAY's HOMEWORK

MODULE

ELLIPSE

Exercise – II (ALMCQ) – COMPLETE

THANK YOU

to all future IITians

PRAYAS 2.0

FOR IIT - JEE 2023

COORDINATE GEOMETRY

HYPERBOLA

LEC – 02

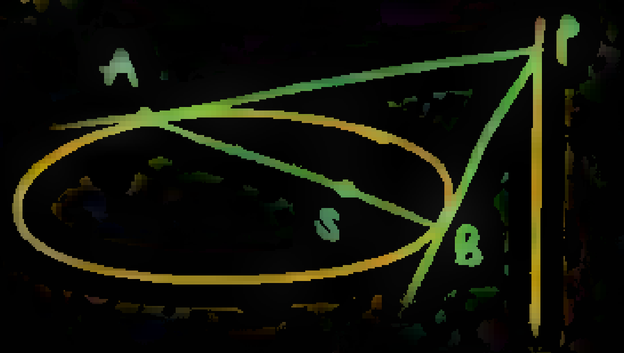
SACHIN JAKHAR

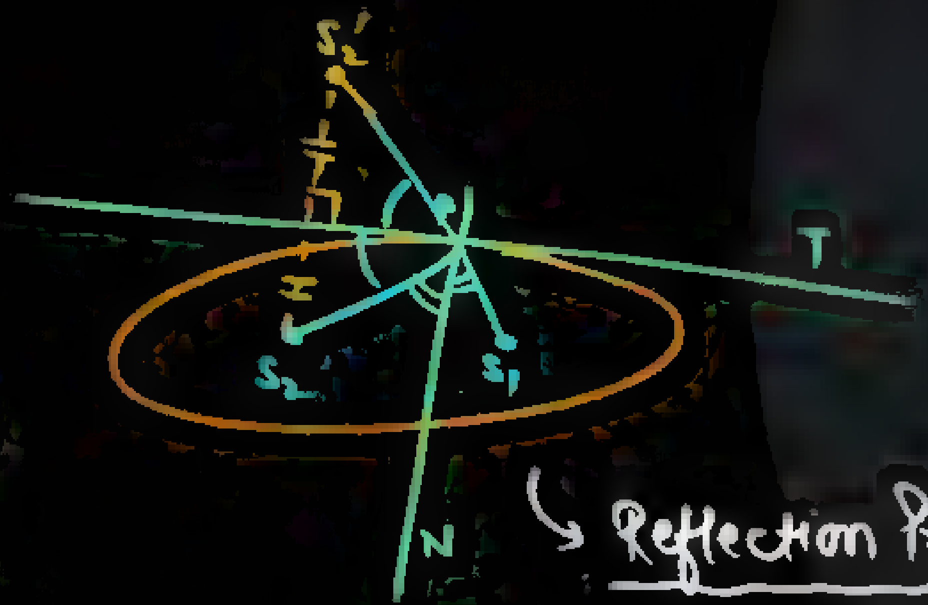
TODAY's GOAL

Auxiliary Circle & Parametric Point
Position of Point w.r.to HB
Line & Hyperbola
Equation of Tangent & Normal

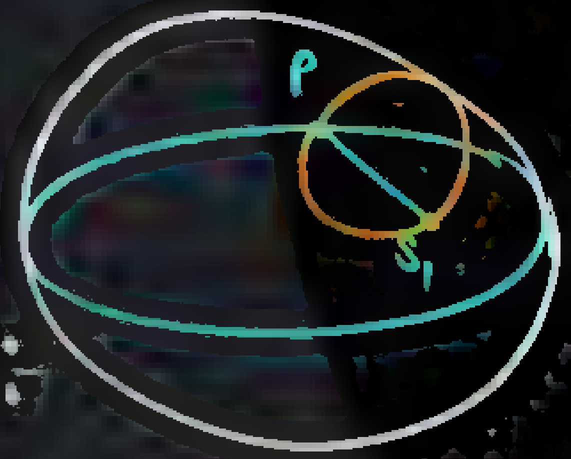
LAST CLASS

Properties of Ellipse:





→ Reflection Prop :



$p_1p_2 = (\text{semi-minor})^2$

A, B → A.C

$Tt \cdot PM = a^2 \cdot b^2$

Introduction to Hyperbola:

$\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ → intersects x-axis but not y-axis

$S(\pm ae, 0)$ $\quad e^2 = 1 + \dfrac{b^2}{a^2}$ $\quad LR = \dfrac{2b^2}{a}$

$\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1}$

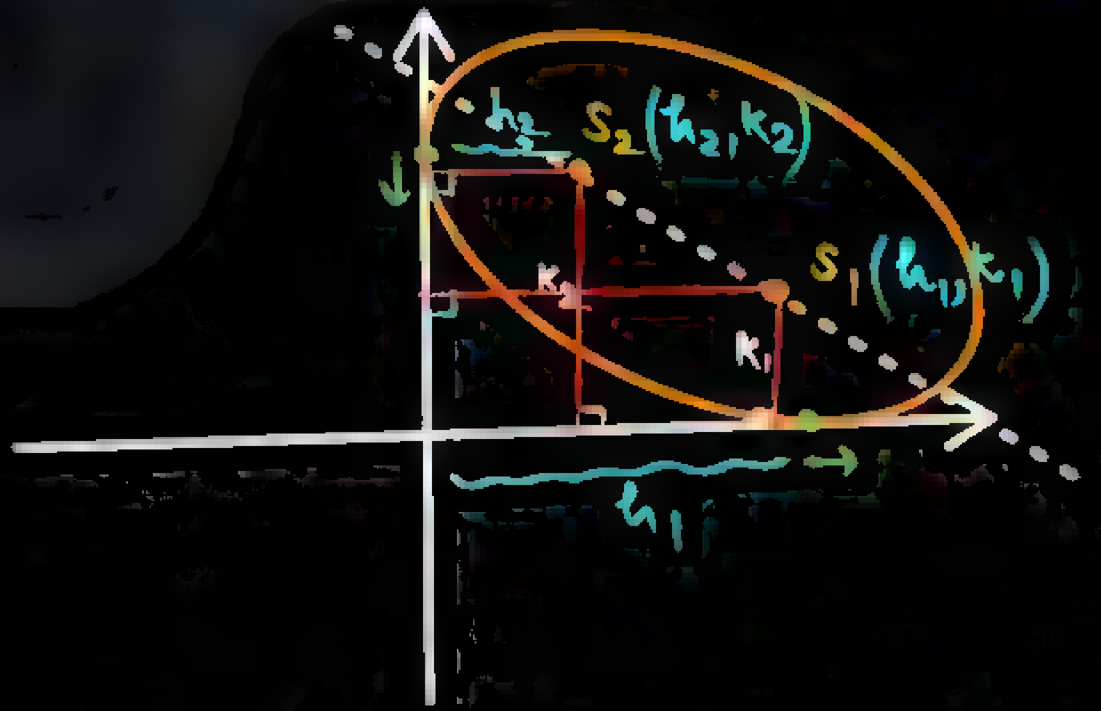
Y-axis → int ✓

X-axis → int ✗

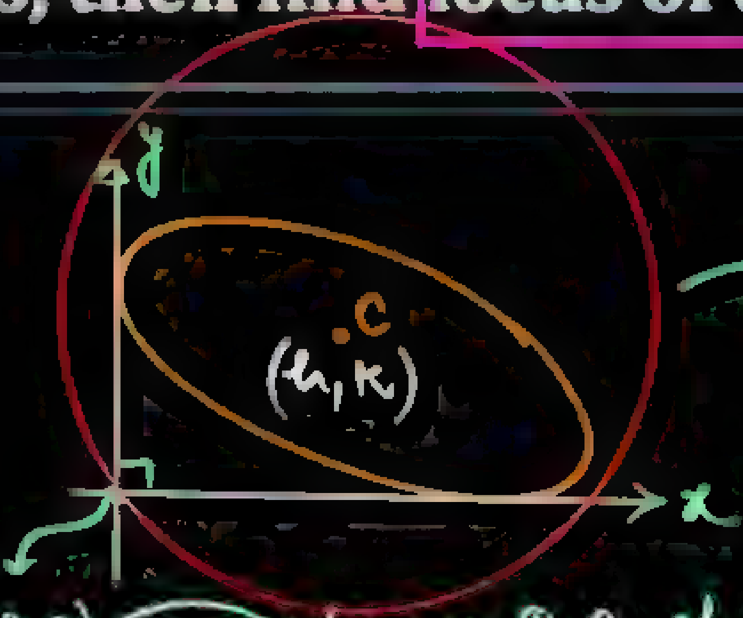
$e^2 = 1 + \dfrac{a^2}{b^2}$ $\quad LR = \dfrac{2a^2}{b}$ $\quad S(0, \pm be)$

Q. An ellipse is with major axis = 2a, minor axis = 2b is sliding between coordinate axes, then find $\boxed{\text{locus of centre}}$ & $\boxed{\text{focii of ellipse}}$?

CHALLENGER



(i)



$(0,0) \longrightarrow$ lies on D.C. of ellipse

eqⁿ: $(x-h)^2 + (y-k)^2 = \left(\sqrt{a^2+b^2}\right)^2$

Pass $(0,0)$

$h^2 + k^2 = a^2 + b^2$

$$\boxed{x^2 + y^2 = a^2 + b^2}$$

(ii) # $S_1S_2^2 = (2ae)^2$

$$\boxed{(h_2-h_1)^2 + (k_2-k_1)^2 = 4a^2e^2}$$

$k_1k_2 = b^2$

$h_1h_2 = b^2$

$h_2 = \frac{b^2}{h_1}$ & $k_2 = \frac{b^2}{k_1}$

$$\boxed{\left(\frac{b^2}{h_1} - h_1\right)^2 + \left(\frac{b^2}{k_1} - k_1\right)^2 = 4a^2\left(1 - \frac{b^2}{a^2}\right)}$$

$h_1 \to x$

$k_1 \to y$

Ex. If $E : \frac{x^2}{16} + \frac{y^2}{b^2} = 1$ & $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ are confocal then find $b^2 = ?$

$$\frac{x^2}{\left(\frac{144}{25}\right)} - \frac{y^2}{\left(\frac{81}{25}\right)} = 1$$

a^2

$$e^2 = 1 + \frac{81/25}{144/25} = 1 + \frac{81}{144}$$

$$e^2 = \frac{225}{144}$$

\# $e = \frac{5}{4} \Leftarrow e = \frac{15}{12}$

$$\text{foci} \equiv \left(\pm \frac{\cancel{12}^{3}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{4}}, 0\right)$$

$$\equiv (\pm 3, 0)$$

\# $e^2 = 1 - \frac{b^2}{16}$

$$\text{foci} \equiv (\pm ae, 0)$$

$$ae = 3$$

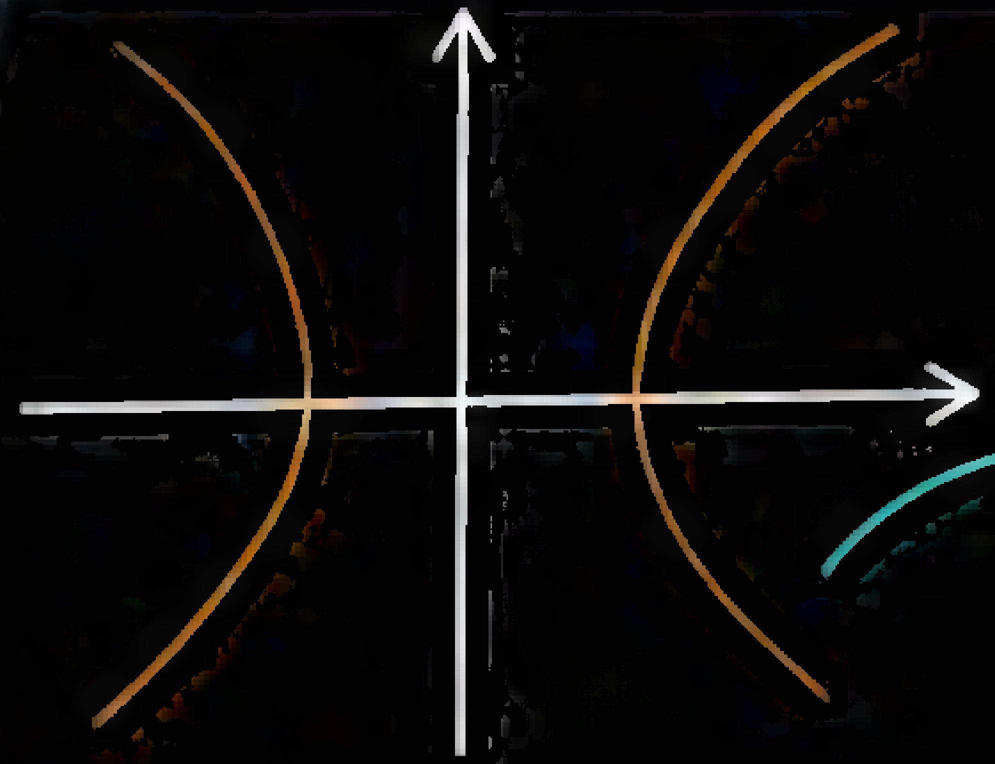
$$4\sqrt{1 - \frac{b^2}{16}} = 3$$

$$\cancel{4}\frac{\sqrt{16 - b^2}}{\cancel{4}} = 3 \Rightarrow 16 - b^2 = 9$$

$$\Uparrow$$

$$\boxed{7 = b^2}$$

Ex. Find the equation of the hyperbola whose eccentricity is $\sqrt{2}$ and the distance between the foci is 16, taking transverse and conjugate axes of the hyperbola as x and y-axes respectively.



$2ae = 16, \quad e = \sqrt{2}$

$2a(\sqrt{2}) = 16$

$a = 4\sqrt{2}$

$e^2 = 1 + \frac{b^2}{a^2}$

$2 = 1 + \frac{b^2}{a^2}$

$1 = \frac{b^2}{a^2}$

$\boxed{a^2 = b^2}$

\# $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$

$\frac{x^2}{32} - \frac{y^2}{32} = 1$ or $\boxed{x^2 - y^2 = 32}$

Ex. Find e of conic whose parametric equation $x = \dfrac{e^t + e^{-t}}{2}$ & $y = \dfrac{e^t - e^{-t}}{3}$, $t \in R$.

$2x = e^t + \dfrac{1}{e^t}$

$\Downarrow$ sq $\Downarrow$

$4x^2 = e^{2t} + \dfrac{1}{e^{2t}} + 2$

$3y = e^t - \dfrac{1}{e^t}$

$\Downarrow$ sq $\Downarrow$

$9y^2 = e^{2t} + \dfrac{1}{e^{2t}} - 2$

$\Downarrow$

$4x^2 = (9y^2 + 2) + 2 \Rightarrow 4x^2 - 9y^2 = 4$

$\dfrac{x^2}{1} - \dfrac{y^2}{\left(\frac{4}{9}\right)} = 1$

$e^2 = 1 + \dfrac{\left(\frac{4}{9}\right)}{1} = \dfrac{13}{9}$

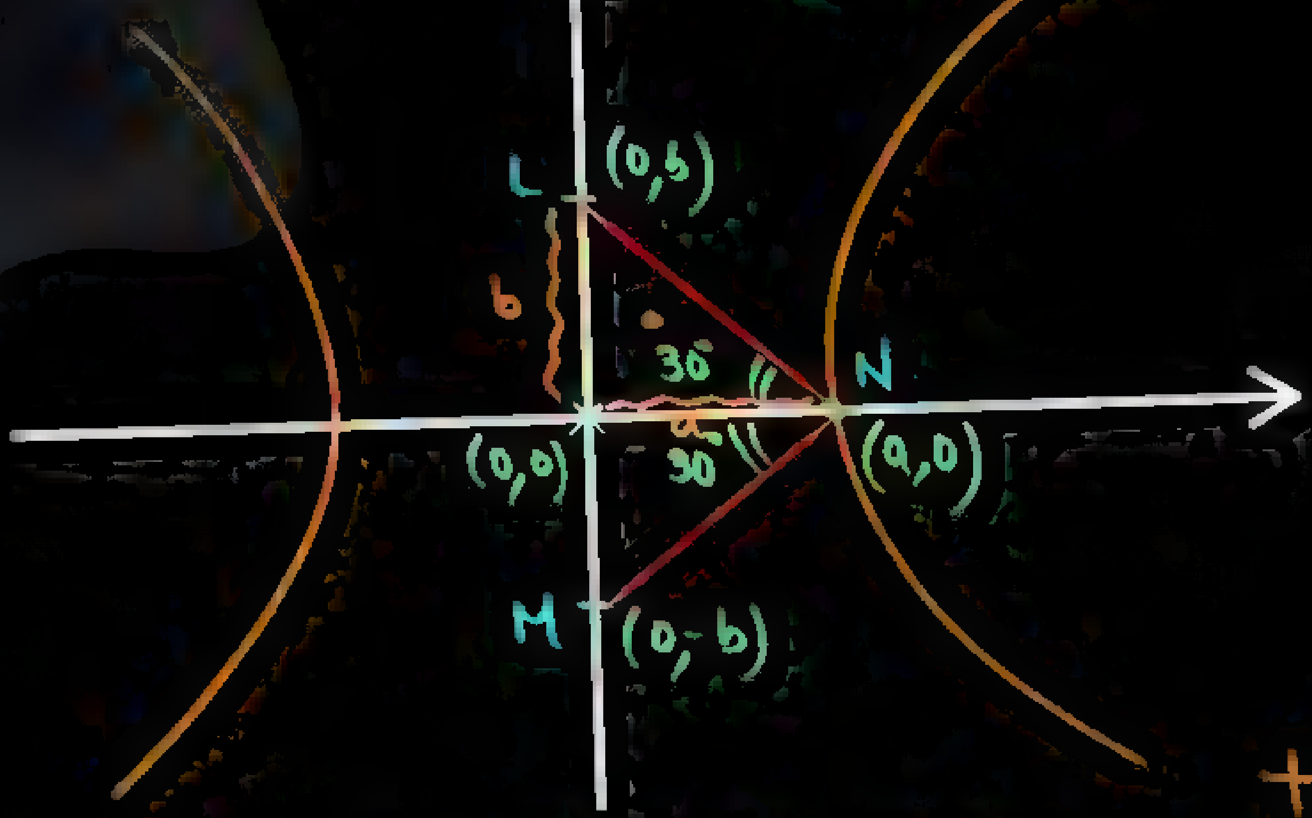
$e = \dfrac{\sqrt{13}}{3}$

Q. Let $H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $\boxed{a > b > 0}$, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

[JEE (Adv.)-2018 (Paper-1)]

List-I		List-II	
A	The length of the conjugate axis of H is (S) $\hookrightarrow 2b = 4$	P	8
B	The eccentricity of H is (R)	Q	$\frac{4}{\sqrt{3}}$
C	The distance between the foci of H is (P) $\hookrightarrow 2ae = 2(2\sqrt{3})\frac{2}{\sqrt{3}} = 8$	R	$\frac{2}{\sqrt{3}}$
D	The length of the latus rectum of H is (Q) $\frac{2b^2}{a} = \frac{2(4)}{2\sqrt{3}}$	S	4

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



$ar(\Delta LMN)=4\sqrt{3}$

$$\frac{1}{\cancel{2}}(\cancel{2}b)a=4\sqrt{3}$$

$$ab=4\sqrt{3}$$

$$a\left(\frac{a}{\sqrt{3}}\right)=4\sqrt{3}\Rightarrow \boxed{a^2=12}$$

$$\tan 30^\circ=\frac{b}{a}$$

$$\frac{1}{\sqrt{3}}=\frac{b}{a}$$

$$\frac{a}{\sqrt{3}}=b \rightarrow b=\frac{2\sqrt{3}}{\sqrt{3}}$$

$$\boxed{b=2}$$

$$\boxed{e^2=1+\frac{4}{12}=1+\frac{1}{3}=\frac{4}{3}}$$

$e=\frac{2}{\sqrt{3}}$

Q. If second degree equations $(x-1)^2+(y-2)^2=\alpha(2x+y-1)^2$ and $\left|\sqrt{(x-1)^2+(y-2)^2}-\sqrt{(x-x_1)^2+(y-y_1)^2}\right| = k^2$ represent same conic then find (x_1, y_1), α & k.

3

($\alpha = 3$; given)

Good Ques

$\boxed{(x-1)^2+(y-2)^2 = 5\alpha\left(\frac{2x+y-1}{\sqrt{5}}\right)^2}$

$\boxed{|PS_1 - PS_2| = k^2}$

$S_1(1,2)$

* $\boxed{S_2(x_1, y_1)}$

$PS^2 = e^2(PM)^2$

Hyperbola

$\boxed{S(1,2)}$, $\boxed{e^2 = 5\alpha}$, $D \equiv \boxed{2x+y-1=0}$

Transverse axis $= k^2 = 2a$

$k^2 = \frac{3\sqrt{3}}{7}$

$e^2 = 15$

$e = \sqrt{15}$

$\left(ae - \frac{a}{e}\right)$

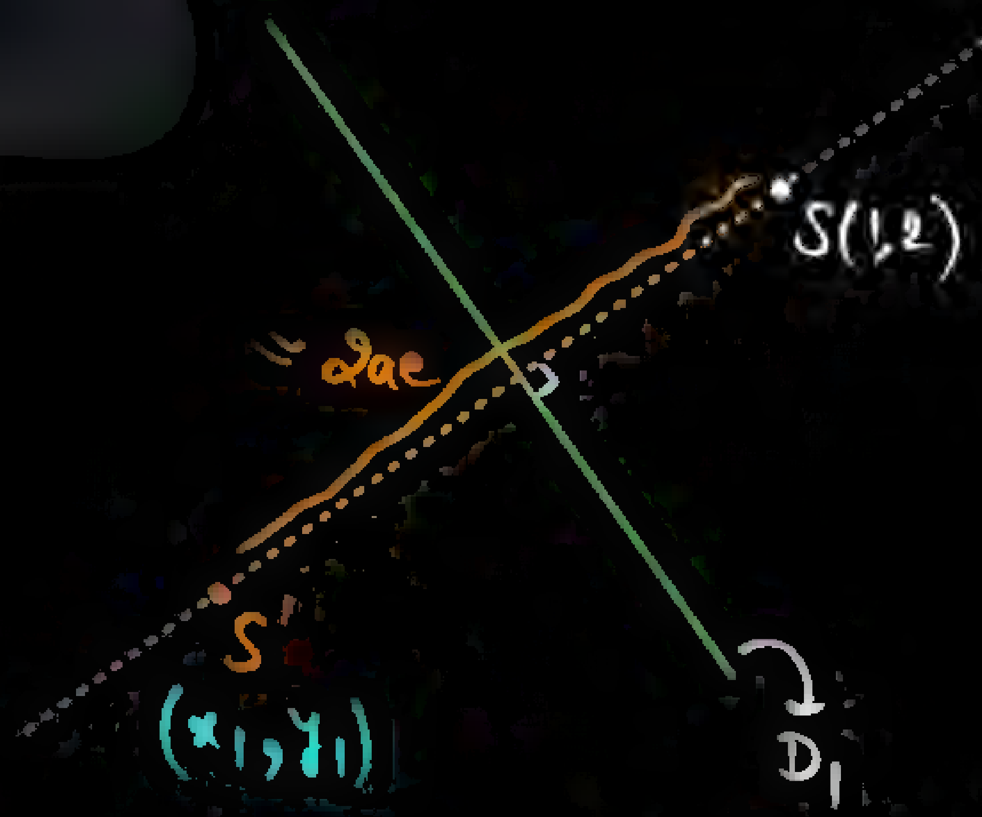
$S(1,2)$

$ae - \frac{a}{e} = \left|\frac{2(1)+2-1}{\sqrt{5}}\right|$

$a\left(\frac{e^2-1}{e}\right) = \frac{3}{\sqrt{5}} \rightsquigarrow a\left(\frac{15-1}{\sqrt{3}\cancel{\sqrt{5}}}\right) = \frac{3}{\cancel{\sqrt{5}}} \Rightarrow 14a = 3\sqrt{3}$

$a = \frac{3\sqrt{3}}{14}$

$m = \tan\theta = \frac{1}{2}$



$a = \frac{3\sqrt{3}}{14}$

$e = \sqrt{15}$

$2x+y-1=0$

$m=-2$

$\frac{x_1-1}{\cos\theta} = \frac{y_1-2}{\sin\theta} = \pm 2ae$

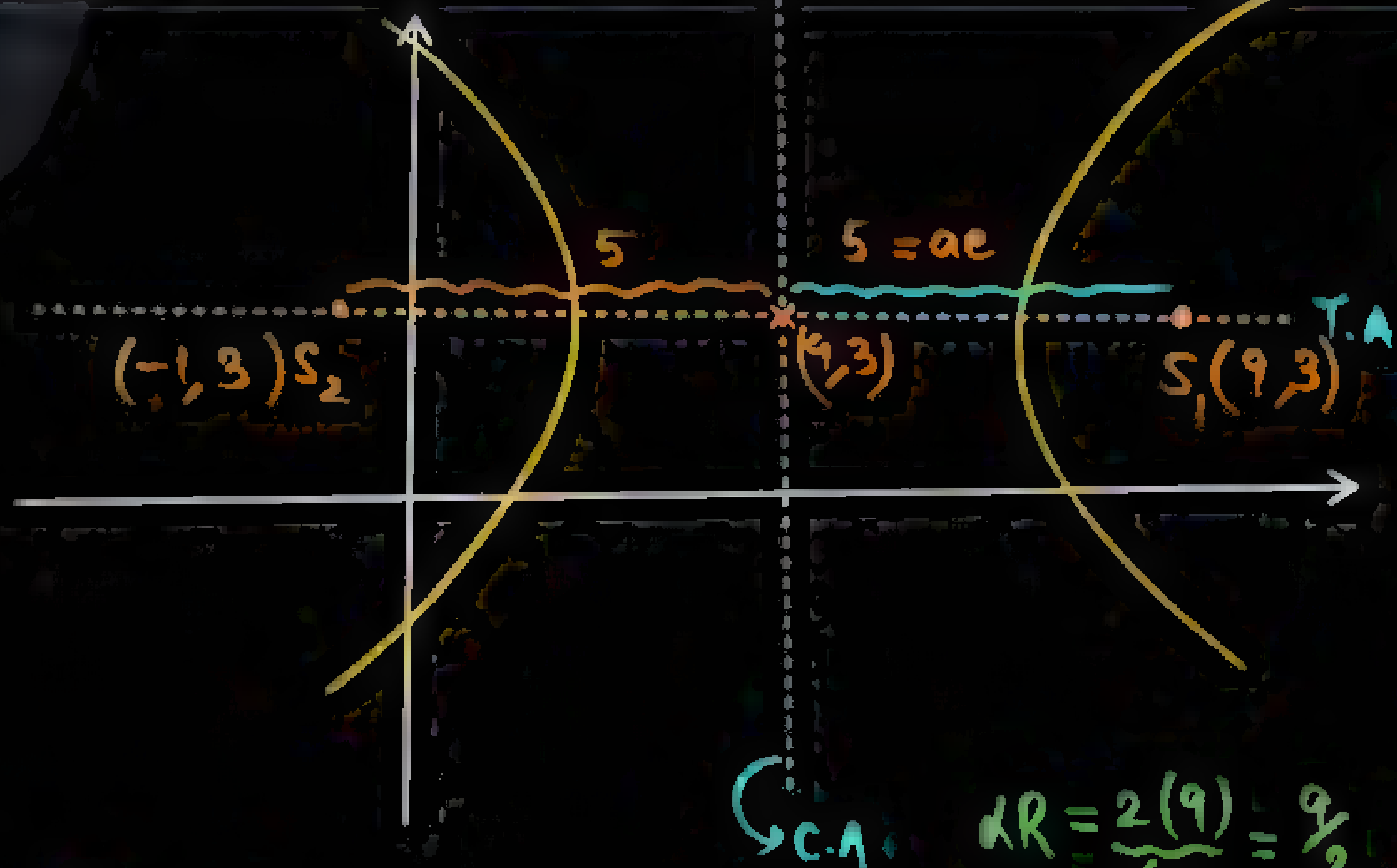
(x_1, y_1)

2 values

find correct one

SHIFTED HYPERBOLA

Ex. Find everything for Hyperbola : $9x^2 - 16y^2 - 72x + 96y - 144 = 0$?



$$9(x^2 - 8x) - 16(y^2 - 6y) = 144$$

$$9(x^2 - 8x + 16 - 16) - 16(y^2 - 6y + 9 - 9) = 144$$

$$9(x-4)^2 - 144 - 16(y-3)^2 + \cancel{144} = \cancel{144}$$

$$9(x-4)^2 - 16(y-3)^2 = (144)$$

$$LR = \frac{2(9)}{4} = \frac{9}{2}$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \xrightarrow{\text{Shift}} \boxed{\frac{(x-4)^2}{16} - \frac{(y-3)^2}{9} = 1}$$

$$e^2 = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow e = \frac{5}{4}$$

$$\Rightarrow \left.\begin{array}{l} a = 4 \\ b = 3 \end{array}\right\} \rightarrow ae = 5$$

EQUILATERAL / RECTANGULAR HYPERBOLA

If length of conjugate axis and transverse axis equal then hyperbola is called as Rectangular/Equilateral hyperbola.

If $\boxed{a=b}$ $\Rightarrow$ $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$a=b$

$\boxed{x^2-y^2=a^2}$ $\rightarrow$ $e^2=1+\frac{a^2}{a^2}$ $\Rightarrow$ $e^2=2$

$\boxed{e=\sqrt{2}}$

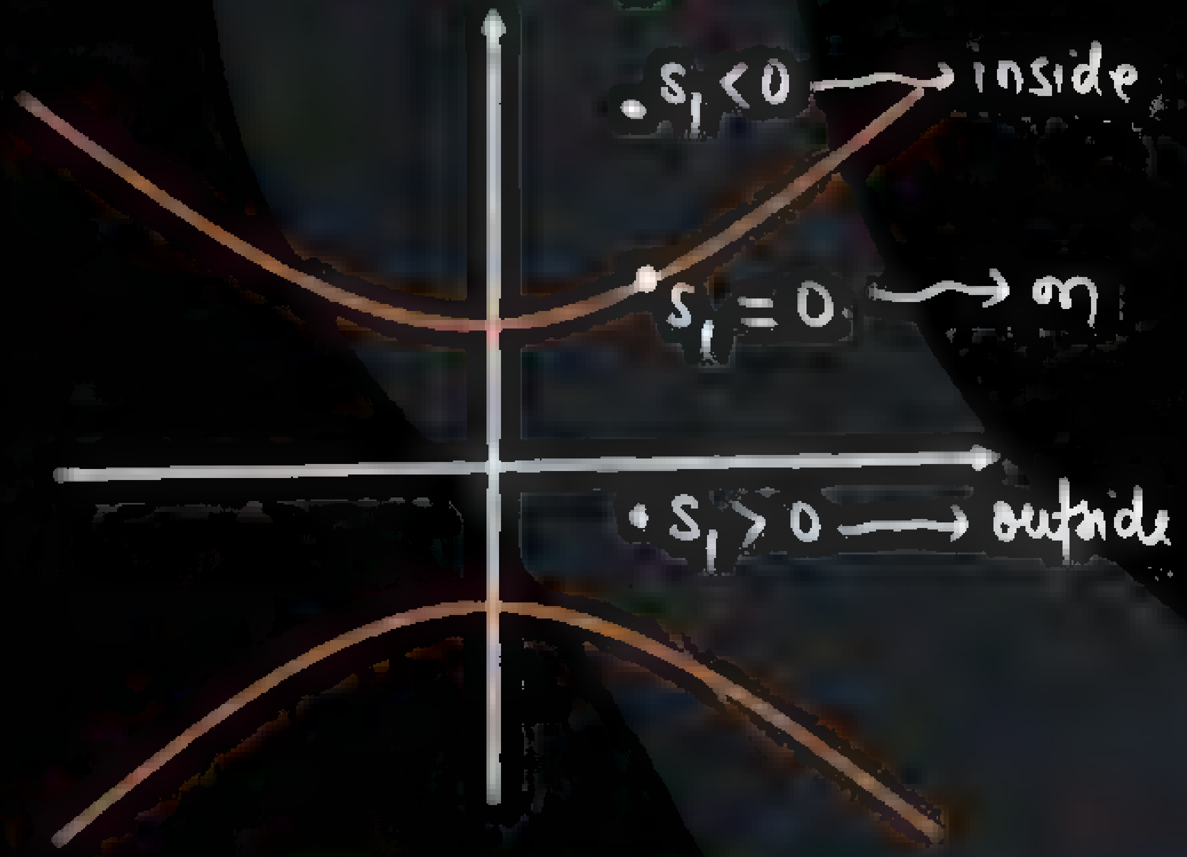
POSITION OF POINT W.R.TO HYPERBOLA

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$$\boxed{S_1=\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)}$$

- outside ⇒ −ve
- on ⇒ zero
- inside ⇒ +ve

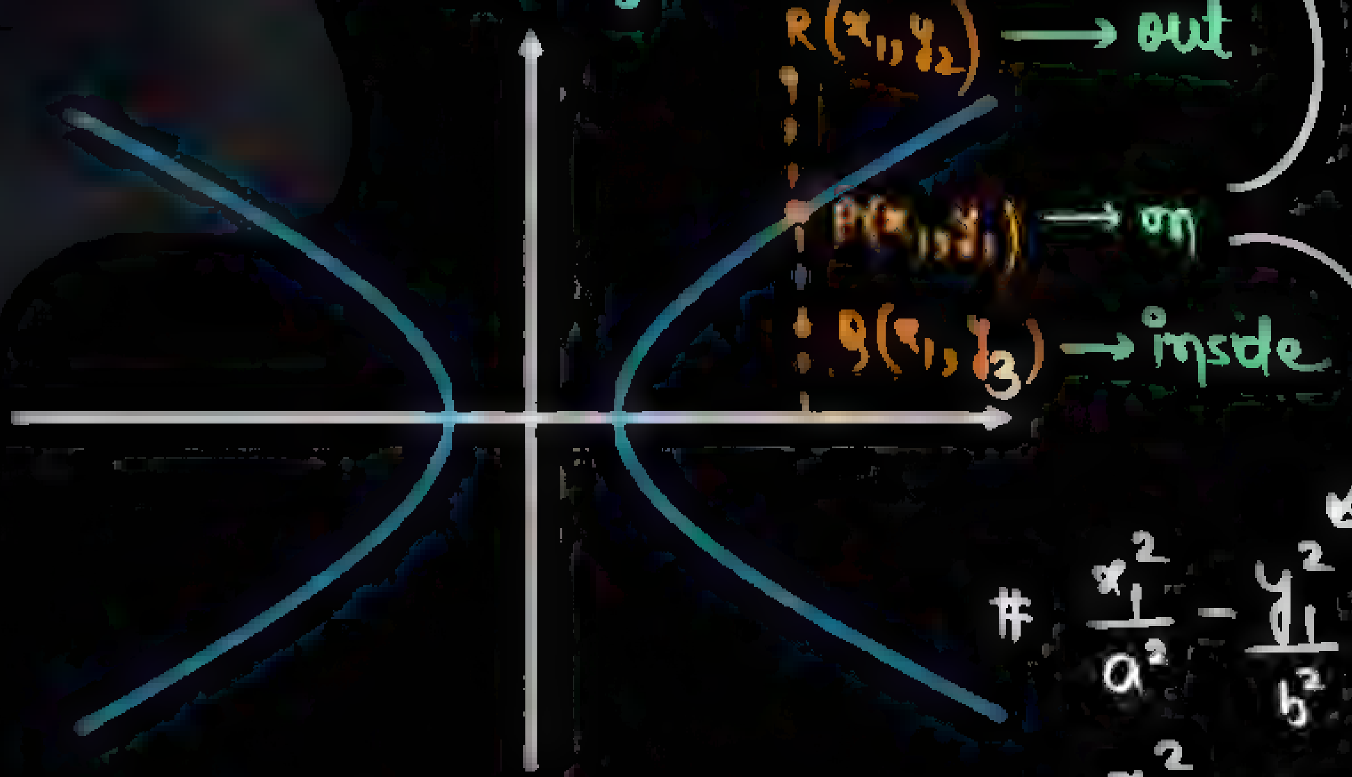
$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=-1$



$$\boxed{S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}+1}$$

- inside → −ve
- on → 0
- outside → +ve

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

$\frac{x_1^2}{a^2} - 1 = \frac{y_1^2}{b^2}$

$|y_2| > |y_1|$

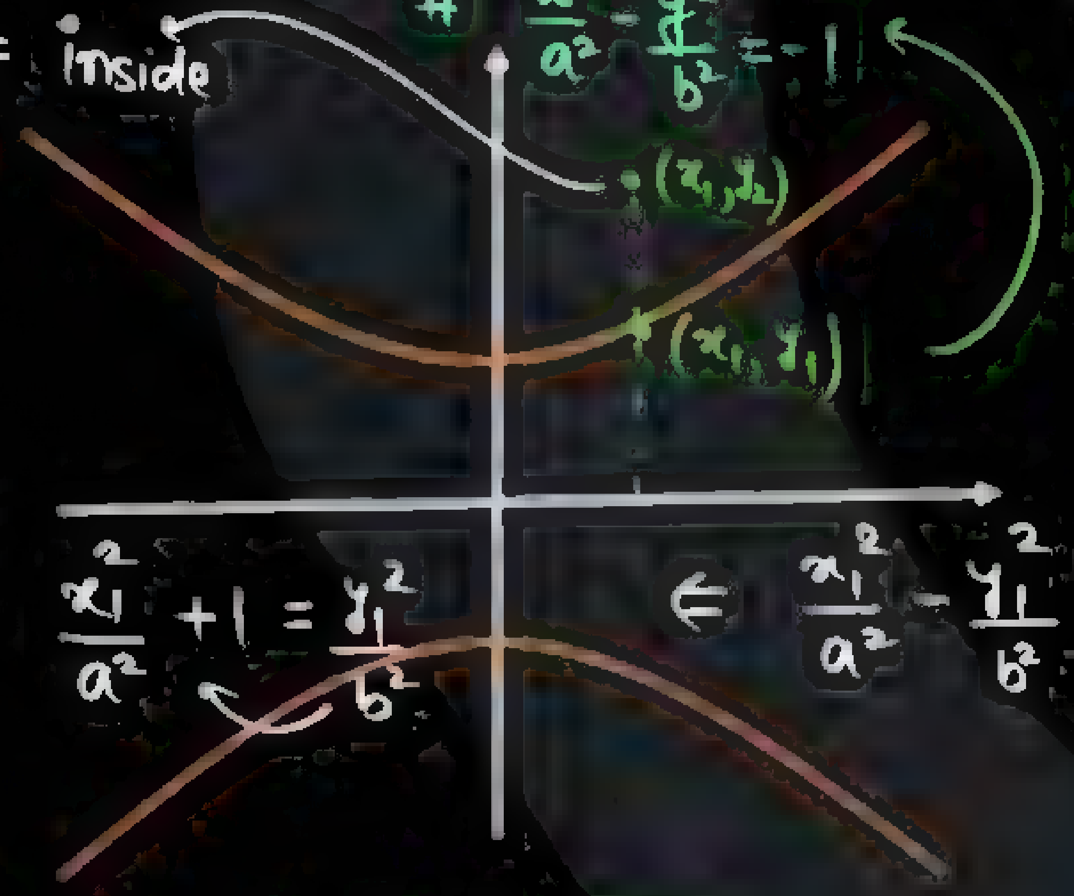
$y_2^2 > y_1^2$

$y_2^2 > b^2\left(\frac{x_1^2}{a^2} - 1\right)$

$\frac{y_2^2}{b^2} > \frac{x_1^2}{a^2} - 1 \Rightarrow \boxed{0 > \left(\frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} - 1\right)}$

inside

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$



$\frac{x_1^2}{a^2} + 1 = \frac{y_1^2}{b^2} \Leftarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = -1$

$|y_2| > |y_1|$

$y_2^2 > y_1^2$

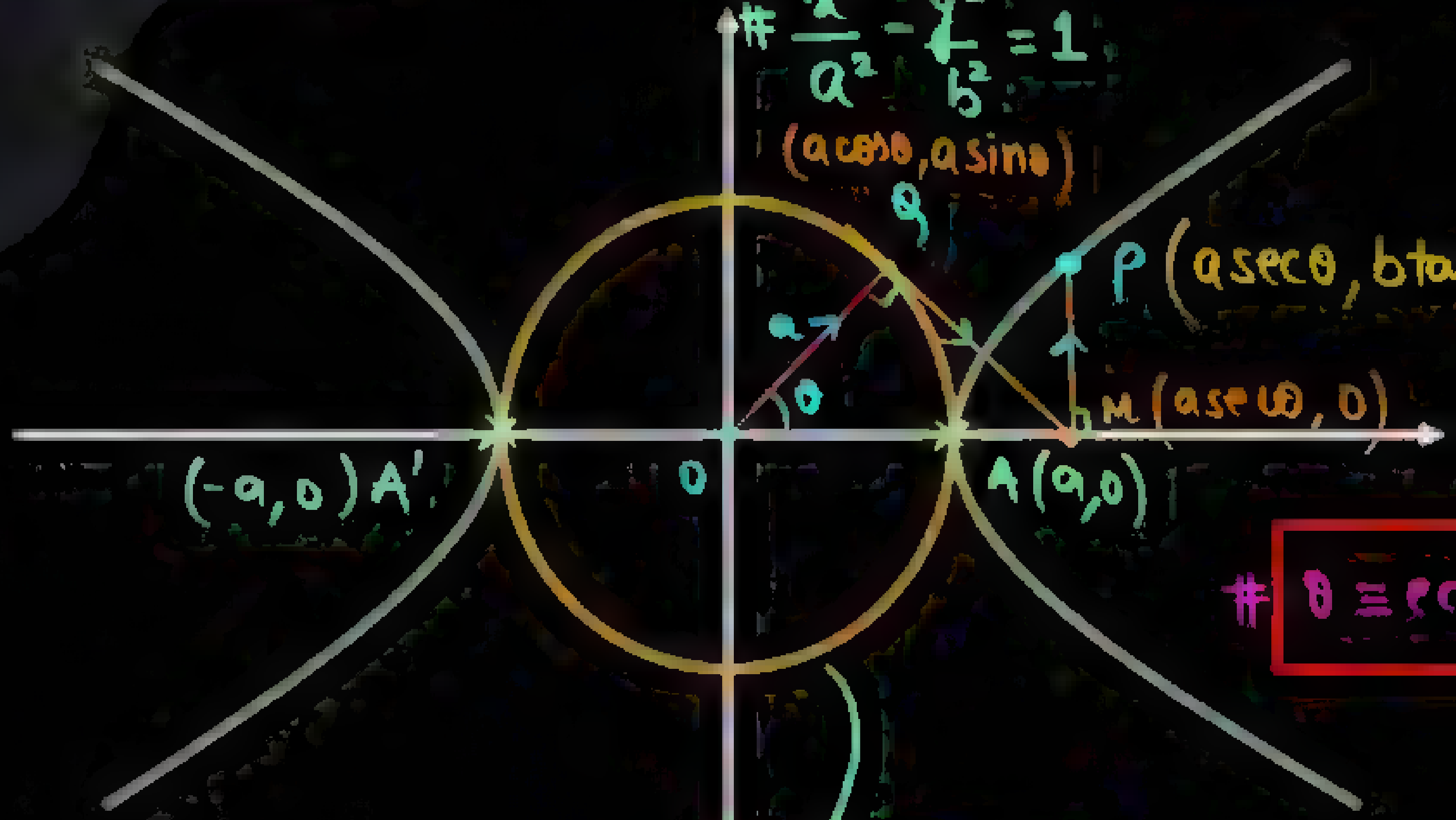
$y_2^2 > b^2\left(\frac{x_1^2}{a^2} + 1\right)$

$\frac{y_2^2}{b^2} > \frac{x_1^2}{a^2} + 1 \Rightarrow 0 > \left(\frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} + 1\right)$

AUXILIARY CIRCLE & PARAMETRIC POINT

Auxiliary Circle: ***Circle with transverse axis as diameter.***

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$\boxed{x^2 + y^2 = a^2}$

$\boxed{\theta \equiv \text{eccentric angle}}$

#

θ	P(asecθ, btanθ)
Quad → I	I
Quad → II	III
Quad → III	II
Quad → IV	IV

Parametric eqⁿ : $\boxed{\begin{aligned} x &= a\sec\theta \\ y &= b\tan\theta \end{aligned}}$

Note :

Hyperbola		Parametric Point
$\boxed{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1}$	$\Rightarrow$	$\boxed{x = a\sec\theta,\ y = b\tan\theta}$
$\boxed{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=-1}$	$\Rightarrow$	$\boxed{x = a\tan\theta,\ y = b\sec\theta}$

LINE & HYPERBOLA

$\boxed{D = 0}$

Line : $\boxed{y = mx + C}$

Condition of Tangency

Hyperbola : $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1} \longrightarrow \boxed{C = \pm\sqrt{a^2m^2 - b^2}}$

Hyperbola : $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1} \longrightarrow \boxed{C = \pm\sqrt{b^2 - a^2m^2}}$

Note: Range of slope:

Line : $\boxed{y = mx + c} \xrightarrow{\text{C.O.T.}} c = \pm\sqrt{a^2m^2 - b^2}$

$\# \ \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

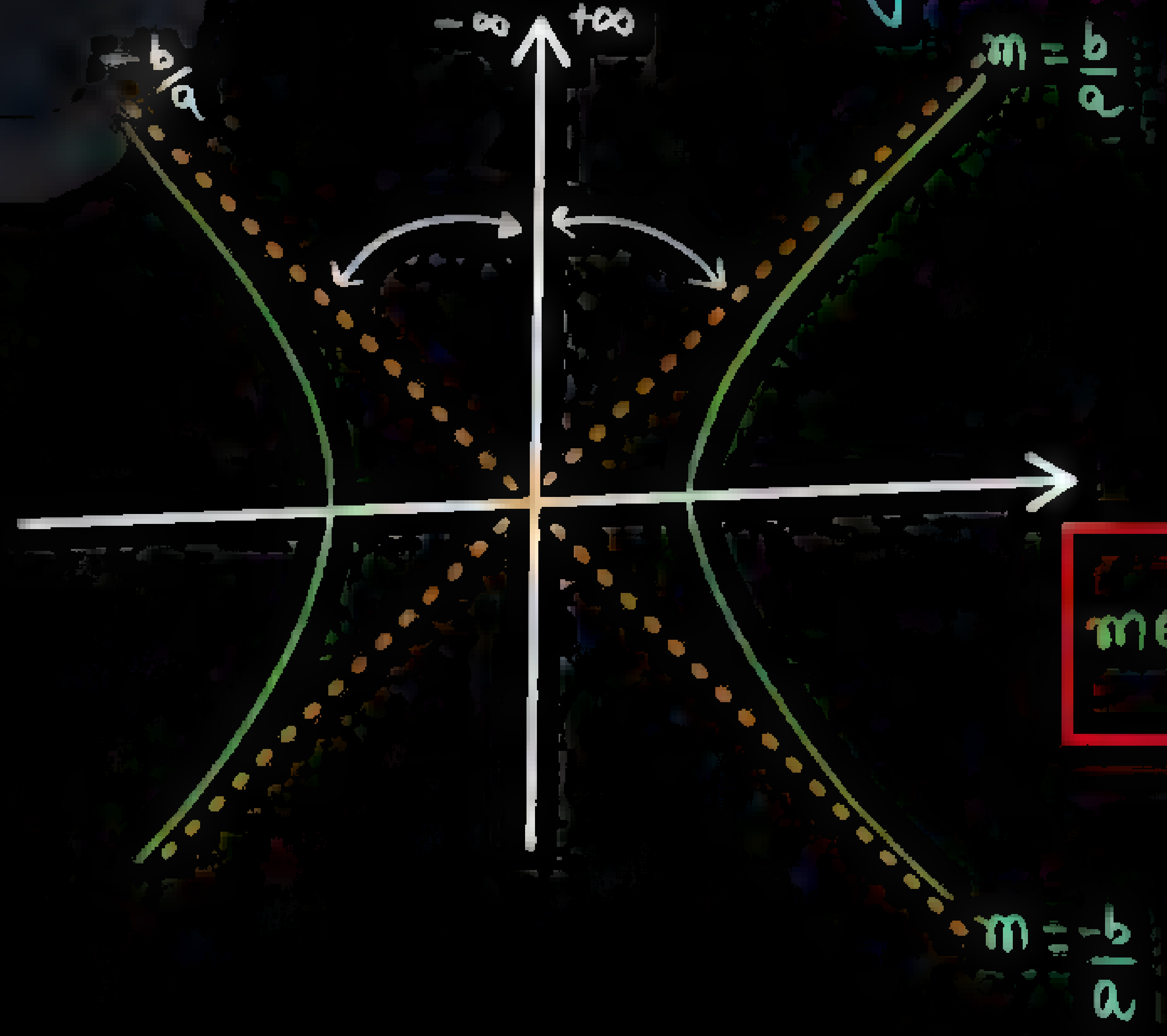
$\# \ a^2m^2 - b^2 \geq 0$

$a^2m^2 \geq b^2$

$m^2 \geq \dfrac{b^2}{a^2}$

$m^2 - \dfrac{b^2}{a^2} \geq 0$

$\left(m - \dfrac{b}{a}\right)\left(m + \dfrac{b}{a}\right) \geq 0$



$\boxed{m \in \left(-\infty, -\dfrac{b}{a}\right] \cup \left[\dfrac{b}{a}, \infty\right)}$

EQUATION OF TANGENT

1. SLOPE FORM: when slope of Tangent is given

Hyperbola	Tangent with slope 'm'
** $\boxed{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1}$	$\boxed{y = mx \pm \sqrt{a^2m^2-b^2}}$
$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=-1 \quad \alpha \quad \boxed{\dfrac{x^2}{(-a^2)}+\dfrac{y^2}{b^2}=1}$ ($a^2\to -a^2$, $b^2\to -b^2$)	$\boxed{y = mx \pm \sqrt{b^2-a^2m^2}}$
$\boxed{\dfrac{(x-\alpha)^2}{a^2}-\dfrac{(y-\beta)^2}{b^2}=1}$ ($x\to x-\alpha$, $y\to y-\beta$)	$\boxed{y-\beta = m(x-\alpha) \pm \sqrt{a^2m^2-b^2}}$ ($x\to x-\alpha$, $y\to y-\beta$)

Cartesian Form : $\boxed{\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1}$

$\begin{cases} x_1 = a\sec\theta \\ y_1 = b\tan\theta \end{cases}$

Parametric Form : $\boxed{\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1}$

Tangent at $P(x_1, y_1)$ on $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$

Tangent from external point :

$\boxed{\text{T in m form}} \rightarrow \boxed{\text{Passes thru point}} \rightarrow \boxed{\text{m mein quad}}$

DIRECTOR CIRCLE

Locus of the point of intersection of perpendicular tangents.

Equation of Director Circle : $\boxed{x^2+y^2=a^2-b^2}$

Circle with same centre & $\boxed{\text{radius} = \sqrt{(\text{Semi T.A})^2-(\text{Semi C.A})^2}}$

Important Note:

1. If $(a<b)$ $\Rightarrow$ No real D.C (No $\perp^r$ tangent exist)

2. If $(a=b)$ $\Rightarrow$ $\boxed{x^2+y^2=0}$ $\Rightarrow$ Point circle (centre of HB)

Rectangular HB

3. If $(a>b)$ $\Rightarrow$ D.C exists

4. For other HB : $\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1}$ $\Rightarrow$ D.C $\Rightarrow$ $\boxed{x^2+y^2=b^2-a^2}$

Q. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, Then which of the following cannot be sides of a right angled triangle?

[JEE (Adv.)-2017 (Paper-1)]

(A) 2a, 4, 1

(B) a, 4, 1

(C) a, 4, 2

(D) 2a, 8, 1

$\boxed{y = 2x + 1}$

$m = 2,\ c = \boxed{1 = \pm\sqrt{a^2(4) - 16}}$

$\Downarrow$

$1 = 4a^2 - 16 \Rightarrow 4a^2 = 17$

$\boxed{a = \frac{\sqrt{17}}{2}}$

$\sqrt{17}, 4, 1$ ✓

$\frac{\sqrt{17}}{2}, 4, 1$ ×

$\frac{\sqrt{17}}{2}, 4, 2$ ×

$\sqrt{17}, 8, 1$ ×

Q. Tangent are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The point of contact of the tangents on the hyperbola are

$a^2 = 9,\ b^2 = 4$

[IIT-JEE-2012 (Paper-1)]

A. $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

B. $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

C. $\left(3\sqrt{3}, -2\sqrt{2}\right)$

D. $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

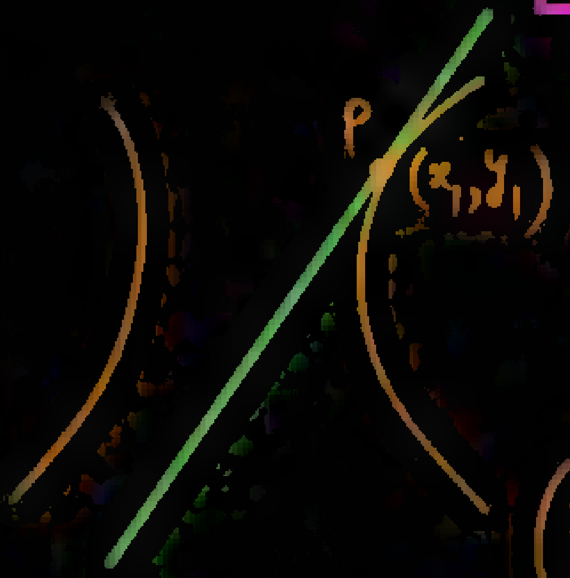
$y = 2x - 1 \Rightarrow m = 2$

eq^n: $y = 2x \pm \sqrt{9(2)^2 - 4}$

$\boxed{y = 2x \pm 4\sqrt{2}} \Rightarrow -2x + y = \pm 4\sqrt{2}$

$\oplus$: $\boxed{-\frac{x}{2\sqrt{2}} + \frac{y}{4\sqrt{2}} = 1}$

$\ominus$: $\boxed{\frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1}$



Tangent at P: $\boxed{\frac{xx_1}{9} - \frac{yy_1}{4} = 1}$

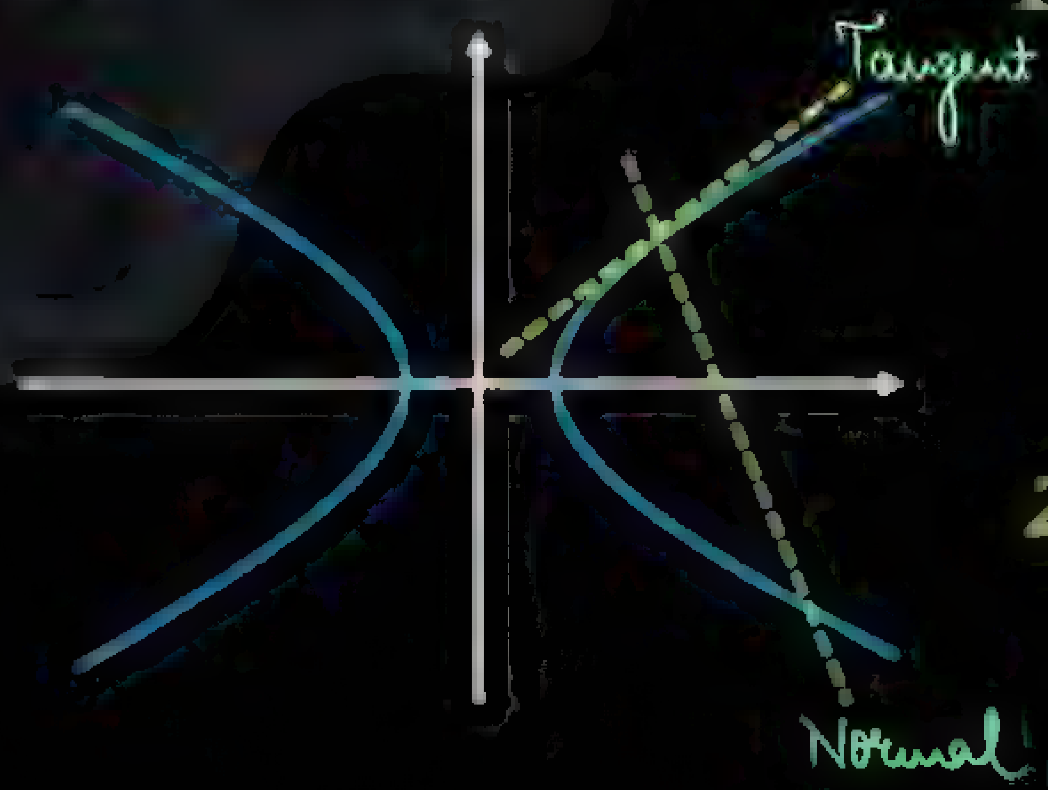
comp

comp

$\left(\frac{-9}{2\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \Leftarrow \frac{x_1}{9} = \frac{-1}{2\sqrt{2}}$ & $\frac{-y_1}{4} = \frac{1}{4\sqrt{2}}$

$\frac{1}{2\sqrt{2}} = \frac{x_1}{9},\ \frac{y_1}{4} = \frac{1}{4\sqrt{2}}$

EQUATION OF NORMAL



1. Normal at Point $\boxed{P(x_1, y_1)}$ when point lies on Hyperbola

*****

$$\boxed{\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2}$$ (valid for both)

2. Parametric Form: when point given in parametric form

$x_1 = a\sec\theta$, $y_1 = b\tan\theta$

$$\boxed{\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2}$$

$x_1 = a\tan\theta$, $y_1 = b\sec\theta$

$$\boxed{\frac{ax}{\tan\theta}+\frac{by}{\sec\theta}=a^2+b^2}$$

3. Slope Form: when slope of Normal is given

$$\boxed{y=mx\pm\frac{(a^2+b^2)m}{\sqrt{a^2-b^2m^2}}}$$

Ex. If line $lx + my - n = 0$ is normal to $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ then show that

$$\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$$

Parametric form:

$$\# \quad \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2+b^2$$

$$\boxed{\frac{a}{(a^2+b^2)\sec\theta}x + \frac{b}{(a^2+b^2)\tan\theta}y = 1}$$

$$lx + my = n$$

$$\boxed{\left(\frac{l}{n}\right)x + \left(\frac{m}{n}\right)y = 1}$$

HHPP

$$\frac{l}{n} = \frac{a}{(a^2+b^2)\sec\theta}$$

$$\frac{m}{n} = \frac{b}{(a^2+b^2)\tan\theta}$$

$$\boxed{\begin{aligned}\sec\theta &= \frac{an}{l(a^2+b^2)}\\ \tan\theta &= \frac{nb}{m(a^2+b^2)}\end{aligned}} \Rightarrow \boxed{S\&S} \Rightarrow \frac{a^2n^2}{l^2(a^2+b^2)^2} - \frac{n^2b^2}{m^2(a^2+b^2)^2} = 1$$

Q. Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is

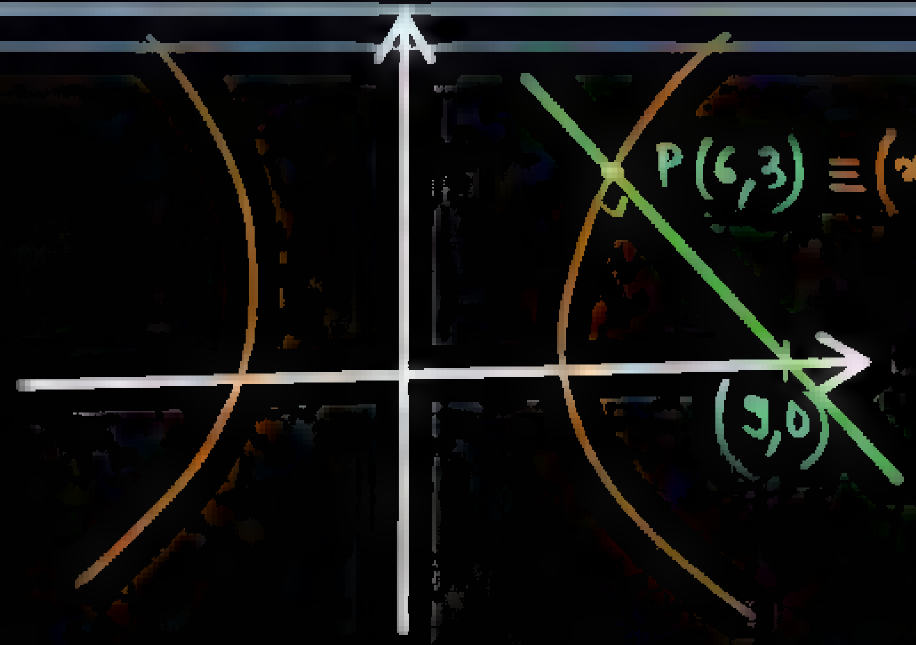
[IIT-JEE-2011 (Paper-2)]

A $\sqrt{\frac{5}{2}}$

B $\sqrt{\frac{3}{2}}$

C $\sqrt{2}$

D $\sqrt{3}$



eqⁿ of N at P : $\boxed{\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2}$

$$\frac{a^2x}{6}+\frac{b^2y}{3}=a^2+b^2$$

(9,0)

$$\frac{a^2(\cancel{9})^{3}}{\cancel{6}_{2}}=a^2+b^2$$

$$\frac{3a^2}{2}-a^2=b^2$$

$$\frac{a^2}{2}=b^2$$

$$\frac{1}{2}=\frac{b^2}{a^2}$$

$$e^2=1+\frac{b^2}{a^2}$$

$$=1+\frac{1}{2}=\frac{3}{2}$$

Q. Let a and b be positive numbers such that a > 1 and b < a. Let P be a point in the first quadrant that lies on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Suppose the tangent to the hyperbola at P passes through the point (1, 0) and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinates axes. Let Δ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following is/are TRUE?

$m = -1$

[JEE (Adv.)-2020 (Paper-2)]

(A) $1 < e < \sqrt{2}$ ✓

(B) $\sqrt{2} < e < 2$

(C) $\Delta = a^4$ ✗

(D) $\Delta = b^4$ ✓

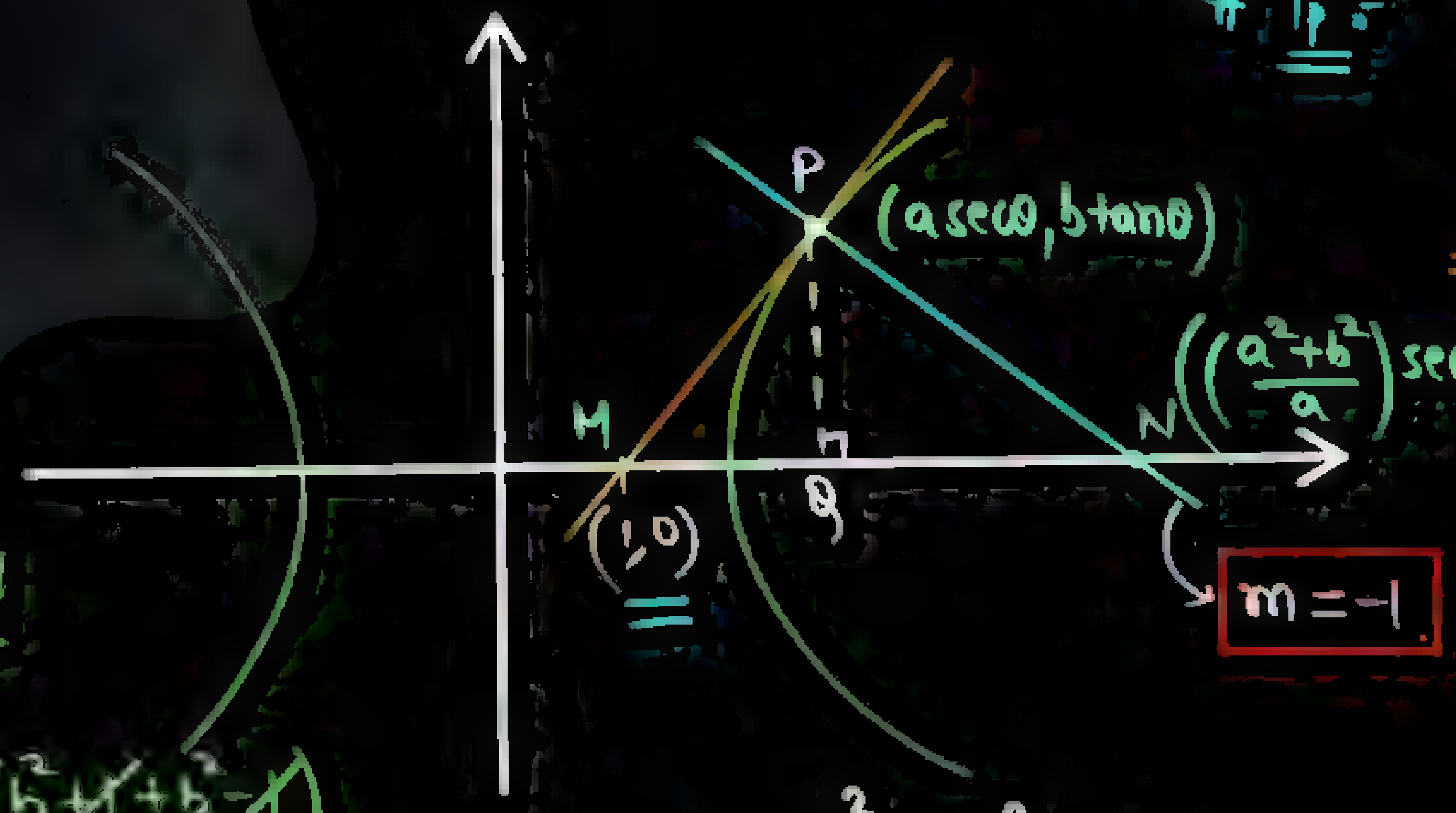
A & D

$e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{a^2-1}{a^2} = 1 + 1 - \frac{1}{a^2} = \left(2 - \frac{1}{a^2}\right)$

$1 < e^2 < 2$

$e^2\big|_{max} \to 2$

$e^2\big|_{min} \to 1$



T_P : $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ → $(1,0)$

$\frac{\sec\theta}{a} = 1 \Rightarrow \boxed{\sec\theta = a}$

$$\frac{a}{\sqrt{a^2-b^2}} = a$$
$$\Downarrow$$
$$1 = \sqrt{a^2-b^2}$$
$$\Downarrow$$
$$1 = a^2-b^2$$
$$1+b^2 = a^2$$
$$b^2 = a^2-1$$

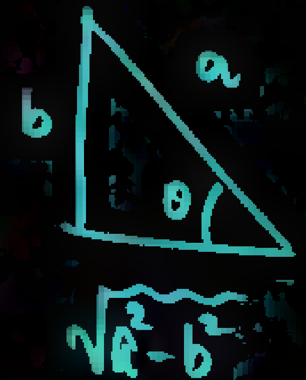
eq^n of N at P :

$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2+b^2$

$$\boxed{m = -1 = -\frac{a}{b}\sin\theta}$$

$y=0$

$$x = \frac{(a^2+b^2)\sec\theta}{a}$$



$$\frac{b}{a} = \sin\theta$$

$$e^2 = 1+\sin^2\theta \qquad e^2 \in [1,2]$$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$A = \frac{1}{2}(MN)(PQ)$$

$$= \frac{1}{2}\left(\frac{a^2+b^2}{a}\sec\theta - 1\right) b\tan\theta$$

$$\frac{b}{2}\left(\left(\frac{a^2+b^2}{\cancel{a}}\right)\frac{\cancel{a}}{\sqrt{a^2-b^2}} - 1\right)\frac{b}{\sqrt{a^2-b^2}}$$

$$\frac{b^2}{2}\left(\frac{a^2+b^2-\overbrace{\sqrt{a^2-b^2}}^{1}}{\underbrace{a^2-b^2}_{1}}\right)$$

$$\frac{b^2}{2}\left(\frac{b^2+\cancel{1}+b^2-\cancel{1}}{1}\right)$$

$$\frac{2b^4}{2}$$

$$b^4$$

Q. Tangents are drawn to the hyperbola $x^2 - y^2 = a^2$ enclosing at an angle of 45°. Show that the locus of their point of intersection is $(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^4$.

H.W

Q. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$, with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A. Then the area of the triangle ABC is

[IIT-JEE-2006 (Paper-2)]

H.W

A. $1 - \sqrt{\frac{2}{3}}$

B. $\sqrt{\frac{3}{2}} - 1$

C. $1 + \sqrt{\frac{2}{3}}$

D. $\sqrt{\frac{3}{2}} + 1$

Q. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

[IIT-JEE-2011 (Paper-1)]

H.W

A The equation of the hyperbola $\frac{x^2}{3} - \frac{y^2}{2} = 1$

B A focus of the hyperbola is (2, 0)

C The eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

D The equation of the hyperbola is $x^2 - 3y^2 = 3$

Q. Show that condition for two concentric ellipse $a_1x^2 + b_1y^2 = 1$ & $a_2x^2 + b_2y^2 = 1$ to intersect ORTHOGONALLY is $\frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{b_1} - \frac{1}{b_2}$?

Ellipse

H.W

TODAY's HOMEWORK

MODULE

HYPERBOLA

Exercise – I (TWQ) – Ques: 1 to 18

Exercise – II (LP) – Ques: 1 to 10

Exercise – III (ALMCQ) – Ques: 1,2,3

THANK YOU

to all future IITians

PRAYAS 2.0

FOR IIT - JEE 2023

COORDINATE GEOMETRY

HYPERBOLA

LEC – 03

SACHIN JAKHAR

TODAY's GOAL

Chord & Focal Chord
Four Important Terms
Asymptotes & its Properties
OP-QP

LAST CLASS

Equation of Tangent:

$$\boxed{y = mx \pm \sqrt{a^2m^2 - b^2}}$$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

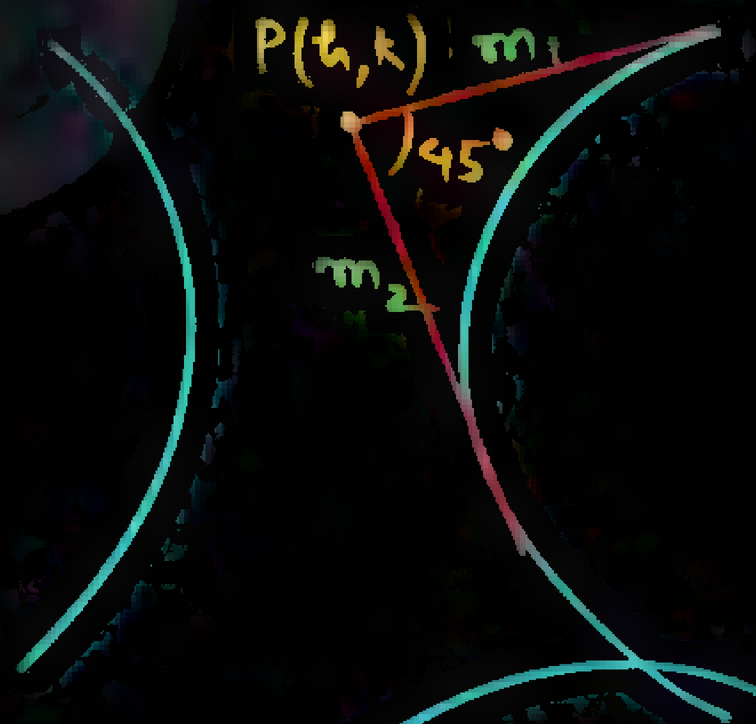
Equation of Normal:

$$\boxed{\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2}$$

Equation of Director Circle:

$$\boxed{x^2 + y^2 = a^2 - b^2} \quad \text{for } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Q. Tangents are drawn to the hyperbola $x^2 - y^2 = a^2$ enclosing at an angle of 45°. Show that the locus of their point of intersection is $(x^2+y^2)^2 + 4a^2(x^2-y^2) = 4a^4$.



Tangent : $y = mx \pm \sqrt{a^2m^2 - a^2}$

Pass (h,k)

$k - mh = \pm\sqrt{a^2m^2 - a^2}$

Sq

$k^2 + m^2h^2 - 2hkm = a^2m^2 - a^2$

$$\boxed{(h^2-a^2)m^2 - (2kh)m + k^2 + a^2 = 0} \quad m_1, m_2$$

$$\tan(45^\circ) = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \Rightarrow 1 = \left|\frac{\dfrac{\sqrt{4k^2h^2 - 4(h^2-a^2)(k^2+a^2)}}{\cancel{h^2-a^2}}}{\dfrac{h^2 - \cancel{a^2} + k^2 + \cancel{a^2}}{\cancel{h^2-a^2}}}\right|$$

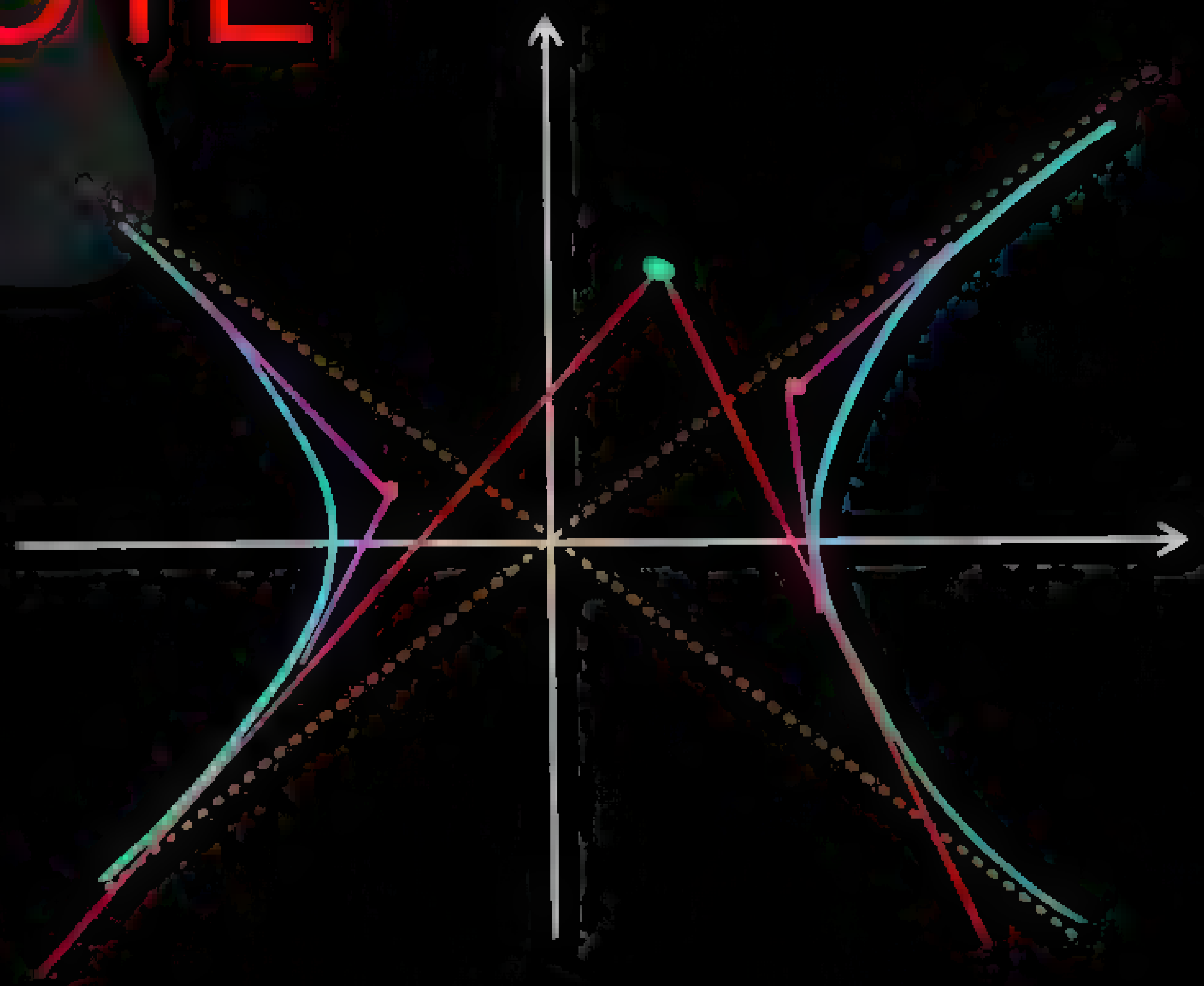
$$\Rightarrow h^2 + k^2 = \sqrt{\cancel{4k^2h^2} - \cancel{4h^2k^2} + 4a^2k^2 - 4a^2h^2 + 4a^4}$$

Sq

$$\boxed{(h^2+k^2)^2 = 4a^2(k^2-h^2) + 4a^4}$$

H.P

NOTE



Q. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$, with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A. Then the area of the triangle ABC is

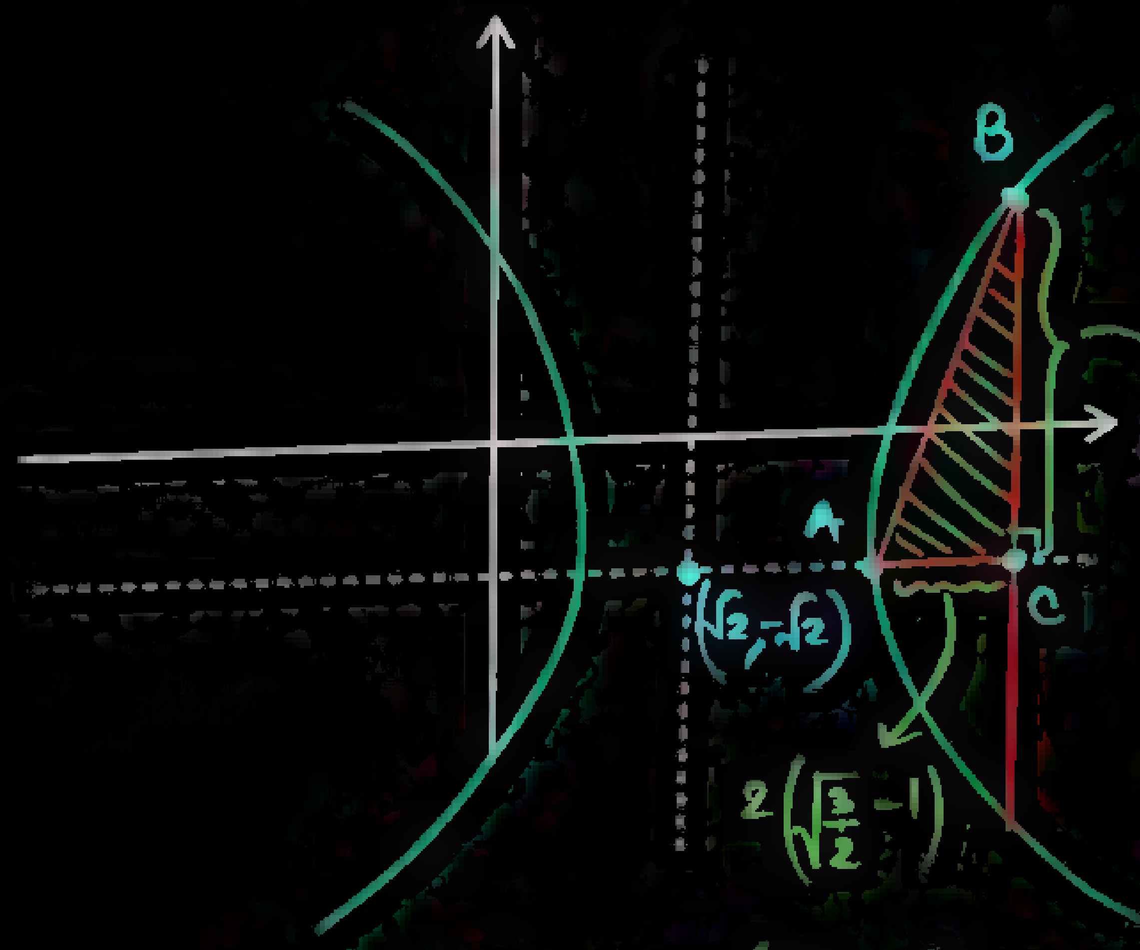
[IIT-JEE-2006 (Paper-2)]

(A) $1 - \sqrt{\dfrac{2}{3}}$

(B) $\sqrt{\dfrac{3}{2}} - 1$

(C) $1 + \sqrt{\dfrac{2}{3}}$

(D) $\sqrt{\dfrac{3}{2}} + 1$



$\frac{b^2}{a} = 1$

area $= \frac{1}{2} \times 2\left(\sqrt{\frac{3}{2}} - 1\right) \times 1$

$x^2 - 2\sqrt{2}x - 2(y^2 + 2\sqrt{2}y) = 6$

$(x^2 - 2\sqrt{2}x + 2 - 2) - 2(y^2 + 2\sqrt{2}y + 2 - 2) = 6$

$(x - \sqrt{2})^2 - 2 - 2(y + \sqrt{2})^2 + 4 = 6$

$(x - \sqrt{2})^2 - 2(y + \sqrt{2})^2 = 4$

$$\boxed{\frac{(x - \sqrt{2})^2}{4} - \frac{(y + \sqrt{2})^2}{2} = 1}$$

$a = 2,\ b = \sqrt{2}$

$e^2 = 1 + \frac{2}{4}$

$e^2 = \frac{3}{2}$

Q. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

$\frac{x^2}{4} + \frac{y^2}{1} = 1 \rightarrow e^2 = 1 - \frac{1}{4} = \frac{3}{4}$

[IIT-JEE-2011 (Paper-1)]

$e_E = \frac{\sqrt{3}}{2} \Rightarrow e_H = \frac{2}{\sqrt{3}}$

(A) The equation of the hyperbola $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) A focus of the hyperbola is (2, 0) ✓

(C) The eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) The equation of the hyperbola is $x^2 - 3y^2 = 3$ ✓

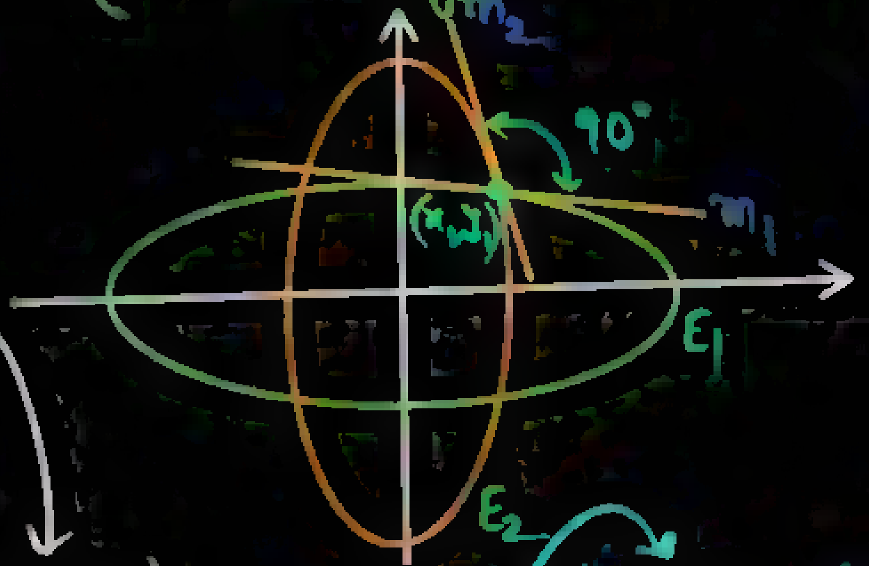
DIBY

Q. Show that condition for two concentric ellipse $a_1x^2+b_1y^2=1$ & $a_2x^2+b_2y^2=1$ to intersect **ORTHOGONALLY** is $\dfrac{1}{a_1}-\dfrac{1}{a_2}=\dfrac{1}{b_1}-\dfrac{1}{b_2}$?

(at Point of intersecⁿ tangents are $\perp^r$)

$E_1:\ \boxed{\dfrac{x^2}{\left(1/a_1\right)}+\dfrac{y^2}{\left(1/b_1\right)}=1}$

$E_2:\ \boxed{\dfrac{x^2}{\left(1/a_2\right)}+\dfrac{y^2}{\left(1/b_2\right)}=1}$



Given : $\boxed{m_1m_2=-1}$

$\left(-\dfrac{a_1x_1}{b_1y_1}\right)\times\left(-\dfrac{a_2x_1}{b_2y_1}\right)=-1$

(x_1,y_1): $a_1x_1^2+b_1y_1^2=1$

(x_1,y_1): $a_2x_1^2+b_2y_1^2=1$

sub: $(a_1-a_2)x_1^2+(b_1-b_2)y_1^2=0$

$(a_1-a_2)x_1^2=-(b_1-b_2)y_1^2$

$\boxed{\dfrac{x_1^2}{y_1^2}=\dfrac{-(b_1-b_2)}{(a_1-a_2)}}$

$\dfrac{a_1a_2}{b_1b_2}\left(\dfrac{x_1^2}{y_1^2}\right)=-1$

$\dfrac{a_1a_2}{b_1b_2}\left(\dfrac{\cancel{-}(b_1-b_2)}{(a_1-a_2)}\right)=\cancel{-}1$

$\dfrac{a_1 a_2}{b_1 b_2}\left(\dfrac{b_1 - b_2}{a_1 - a_2}\right) = 1$

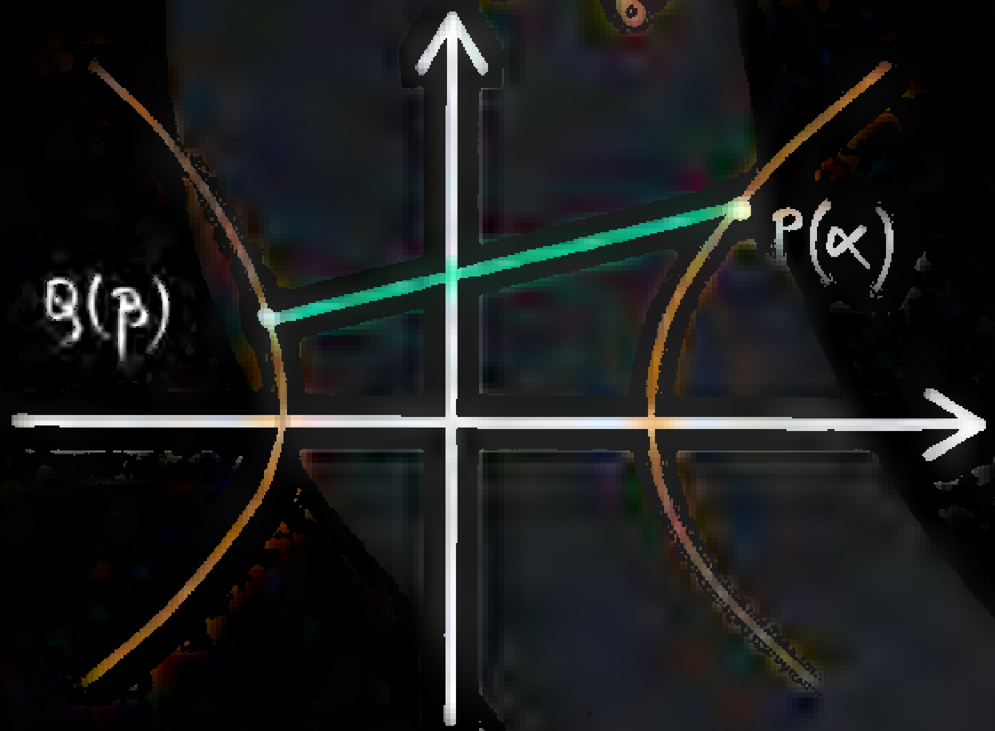
$$\frac{b_1 - b_2}{b_1 b_2} = \frac{a_1 - a_2}{a_1 a_2}$$

$$\boxed{\frac{1}{b_2} - \frac{1}{b_1} = \frac{1}{a_2} - \frac{1}{a_1}}$$

HHPP

CHORD & FOCAL CHORD

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Equation of chord joining P(α) & Q(β)

$$\boxed{\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)}$$

If PQ is focal chord passing then $S_1(ae, 0)$

for focal chord

$$\boxed{\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e}}$$

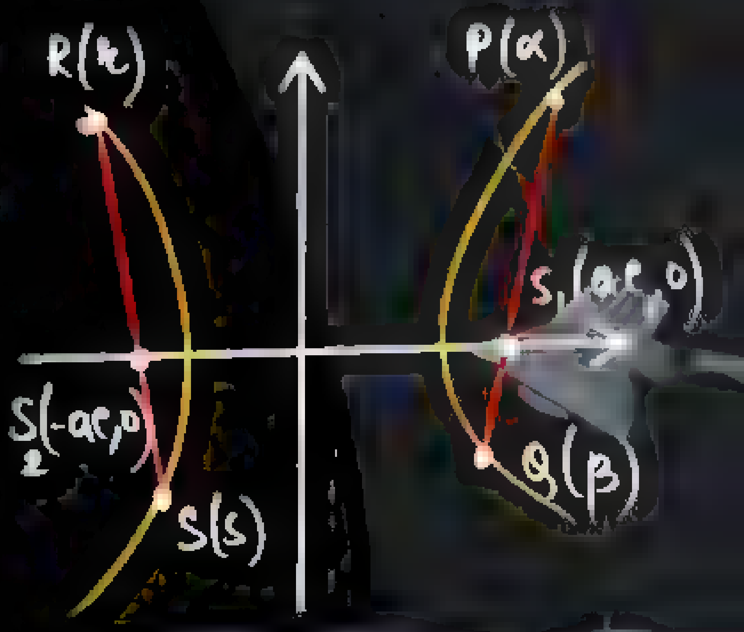
Similarly for another focal chord AB

$R(r)$ & $S(s)$
$\Downarrow$
$S_2(-ae, 0)$

$$\tan\frac{r}{2}\tan\frac{s}{2} = \frac{1+e}{1-e}$$

Same as ellipse

$$\boxed{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{r}{2}\tan\frac{s}{2} = 1}$$



FOUR IMPORTANT TERMS

1. ***Chord of Contact:*** # $\boxed{T_1 = 0}$

2. ***Chord with given midpoint:***

$\boxed{T_1 = S_1}$

3. ***Pair of Tangents:*** $\boxed{T_1^2 = S S_1}$

4. ***Pole & Polar:*** Polar $\Rightarrow$ $\boxed{T_1 = 0}$

NOTE :

$\boxed{ax^2+bx+c=0}$ → ∞, α

If one root is at infinity ⇓ $\boxed{a=0}$

$x \to \frac{1}{x}$

$\frac{a}{x^2}+\frac{b}{x}+c=0$

$a+bx+cx^2=0$

Root ↓ Reciprocal

$\boxed{cx^2+bx+a=0}$ → 0, $\frac{1}{\alpha}$

⇓

$\text{cond}^n = \boxed{a=0}$

$\frac{1}{\infty} \Rightarrow 0$

$\boxed{ax^2+bx+c=0}$ → ∞, ∞

Reciprocal

$x \to \frac{1}{x}$

$\boxed{cx^2+bx+a=0}$ → 0, 0

⇓

$a=0,\ b=0$

Both root at infinity ⇒ $\boxed{0x^2+0x+c=0}$

non zero

$\boxed{\text{coeff. of } x^2 = \text{coeff. of } x = 0}$

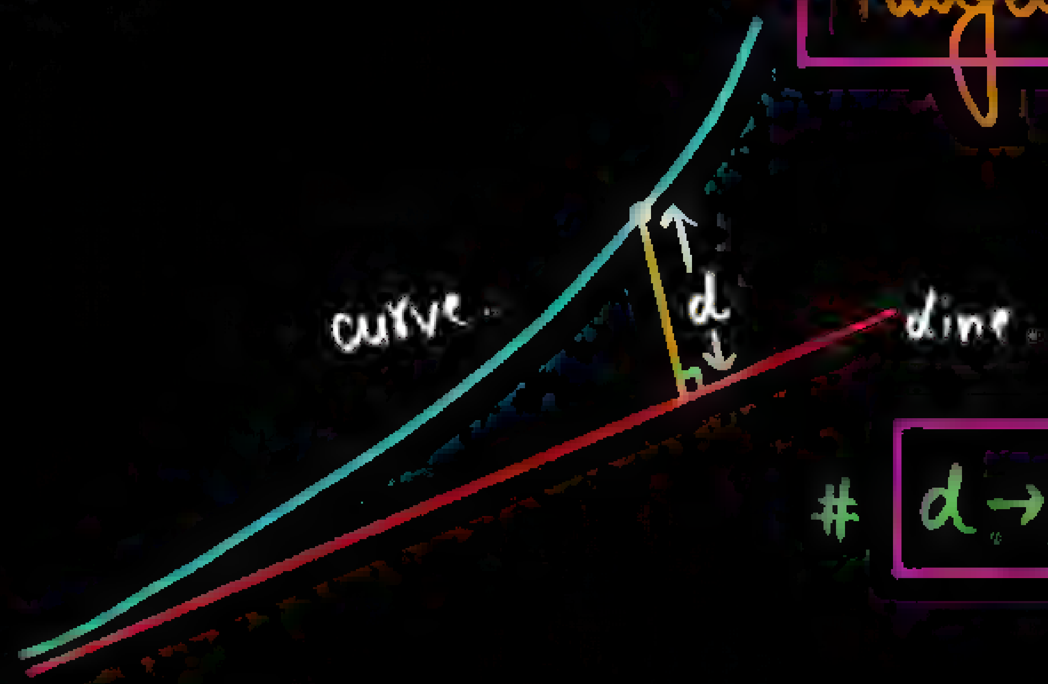
& $\boxed{\text{constant term} \neq 0}$

Asymptotes :

Line which touches the curve at '∞'

or

Tangent at '∞'



\# $\boxed{d \to 0} \Rightarrow$ Line $\to$ $\boxed{\text{Asymptote}}$

ASYMPTOTES

HB : $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ & line : $\boxed{y = mx + c}$

Solve : $\dfrac{x^2}{a^2} - \dfrac{(mx+c)^2}{b^2} = 1$

$b^2x^2 - a^2m^2x^2 - a^2c^2 - 2cma^2x - a^2b^2 = 0$

$$\boxed{(b^2 - a^2m^2)x^2 - (2cma^2)x - (a^2c^2 + a^2b^2) = 0}$$

cond^n : $\boxed{b^2 - a^2m^2 = 0}$ & $\boxed{-2cma^2 = 0}$ ∞ ∞

$\boxed{-a^2(c^2+b^2) \neq 0}$

$\Downarrow$

$\dfrac{b^2}{a^2} = m^2$

$\boxed{m = \pm\dfrac{b}{a}}$ $\boxed{c = 0}$ $-a^2b^2$

line : $y = mx + c \Rightarrow$ Asymp $\Rightarrow$ $\boxed{y = \left(\pm\dfrac{b}{a}\right)x}$ ✓

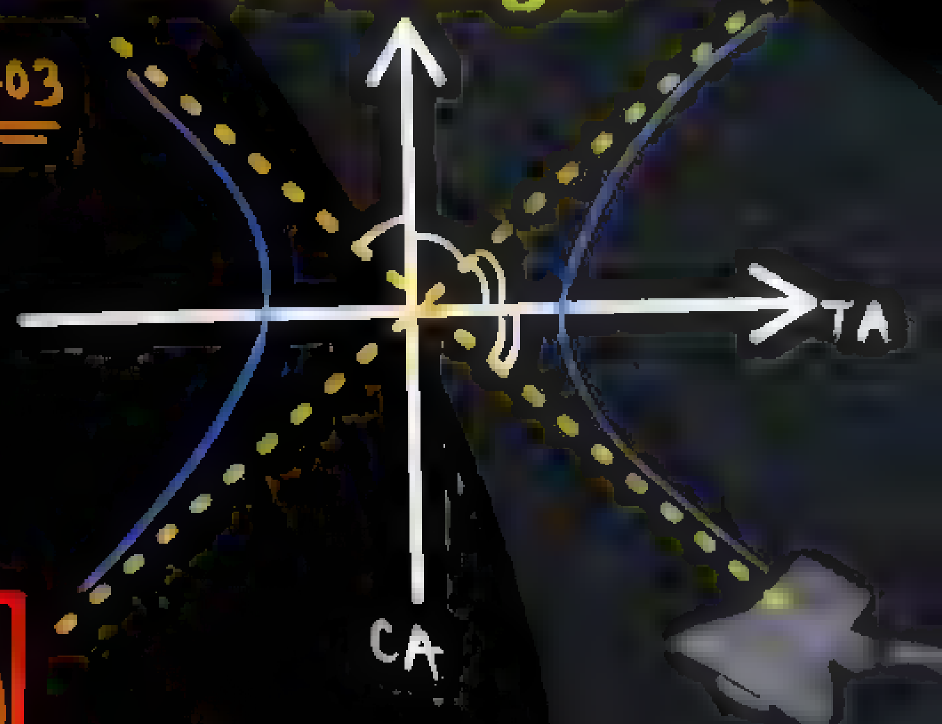
PROPERTIES OF ASYMPTOTES

Property-01: $\boxed{\text{Hyperbola \& Conjugate Hyperbola have same pair of asymptotes.}}$

Property-02: Equation of Hyperbola, Conjugate Hyperbola & pair of Asymptotes only differs in constant part.

Property-03: Asymptotes passes through centre of HB and T.A. & C.A. are angle bisectors of angle between asymptotes.

P-03



P-02 :

HB : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

CHB : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

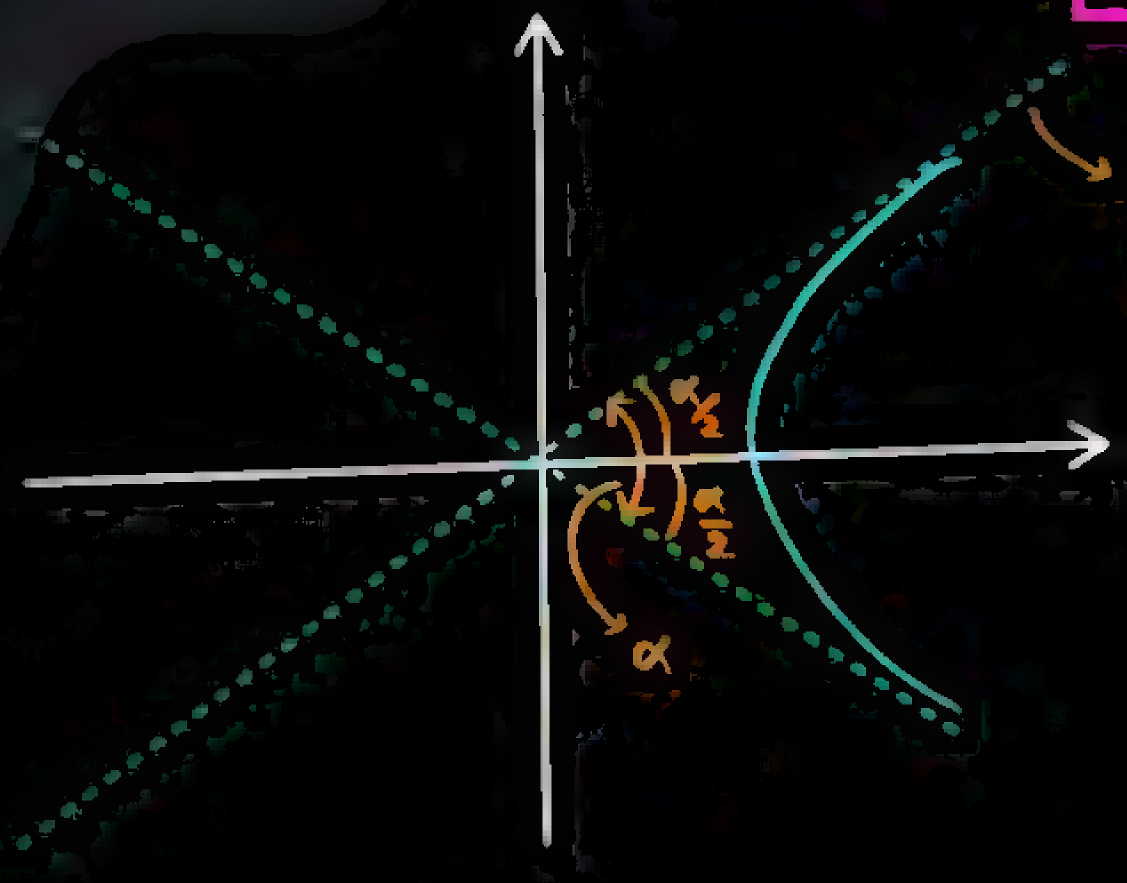
P.O.A : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Generalise : (valid for every type of HB)

$$\boxed{(\text{eq}^n\ \text{HB} + \text{eq}^n\ \text{CHB}) = 2(\text{eq}^n\ \text{P.O. Asym.})}$$

***

Property-04: If angle between asymptotes is 'α' then eccentricity $\boxed{(e) = \sec\left(\frac{\alpha}{2}\right)}$ ***



$y = \frac{b}{a}x$

$$m = \frac{b}{a} = \tan\frac{\alpha}{2}$$

$$e^2 = 1 + \frac{b^2}{a^2} = 1 + \tan^2\frac{\alpha}{2}$$

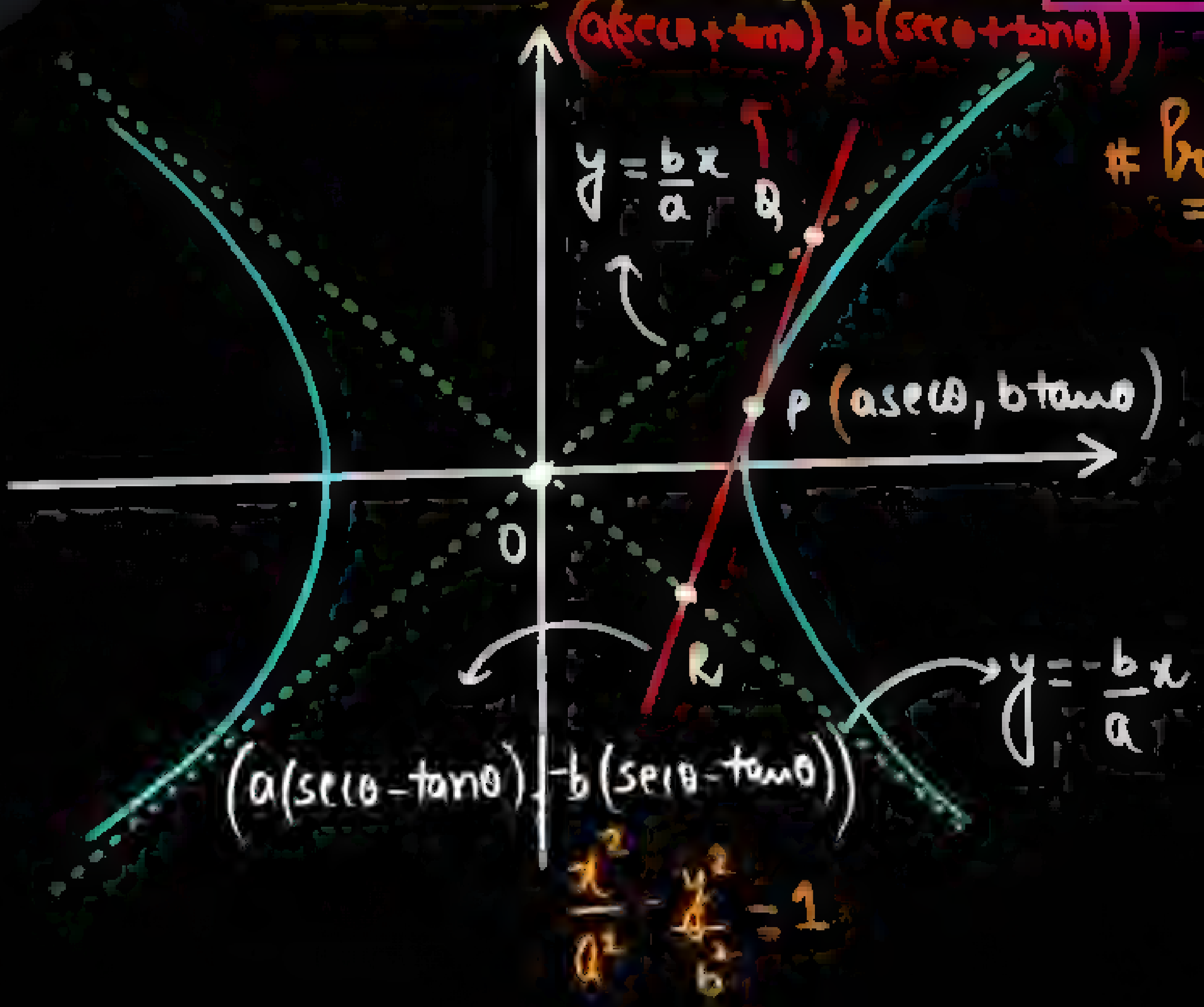
$$e^2 = \sec^2\frac{\alpha}{2}$$

$$\boxed{e = \sec\frac{\alpha}{2}}$$

Property-05:

(i) *Portion of tangent intercepted between pair of asymptotes is bisected at point of contact.* (or midpoint of Q & R is P)

(ii) *Area of triangle formed by any tangent & pair of asymptotes is always constant & is equals to 'ab'* (or area $(\Delta OQR) = ab$)



Proof: T_P: $\boxed{\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1}$

Q solve $\boxed{y=\frac{b}{a}x}$

R solve $\boxed{y=-\frac{b}{a}x}$

$\Rightarrow \frac{x\sec\theta}{a}-\frac{\cancel{b}x}{a}\frac{\tan\theta}{\cancel{b}}=1$

$\frac{x}{a}(\sec\theta-\tan\theta)=1$

$x=\frac{a}{\sec\theta-\tan\theta}=a(\sec\theta+\tan\theta)$

$(a(s+t),\ b(s+t))$

$$ar(\Delta OQR) = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a(s+t) & b(s+t) & 1 \\ a(s-t) & -b(s-t) & 1 \end{vmatrix} = \frac{1}{2}\left| -ab(1) - ab(1) \right|$$

$(0,0)$ $\quad (a(s-t),\ -b(s-t))$

$$= \frac{|-2ab|}{2}$$

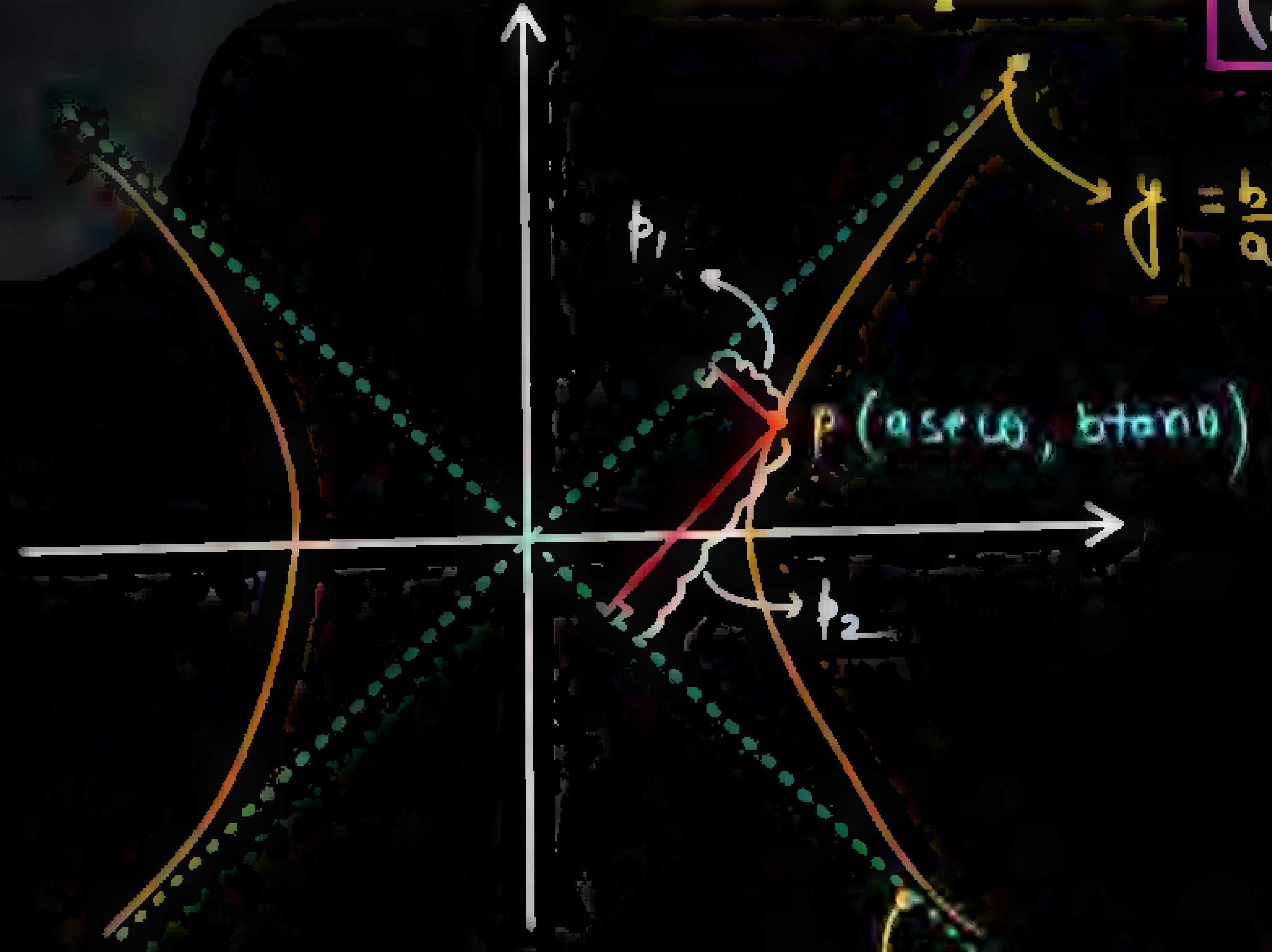
$\Delta = ab$

S :

Summary
Sachin
Sir
Sc

$ar(OQR) = ab$

Property-06: From any point on Hyperbola product of lengths of perpendicular drawn on asymptotes is always constant and is equals to $\boxed{\left(\frac{a^2b^2}{a^2+b^2}\right) = p_1p_2}$



$y = \frac{b}{a}x \Rightarrow \boxed{bx - ay = 0}$

$y = -\frac{b}{a}x \Rightarrow \boxed{bx + ay = 0}$

$$p_1 = \left|\frac{ba\sec\theta - ab\tan\theta}{\sqrt{a^2+b^2}}\right| \qquad ab(\sec\theta - \tan\theta)$$

$$p_2 = \left|\frac{ba\sec\theta + ab\tan\theta}{\sqrt{a^2+b^2}}\right| \qquad ab(\sec\theta + \tan\theta)$$

$$\Rightarrow p_1p_2 = \frac{a^2b^2}{a^2+b^2}$$

H.P.

NOTE :

eq^n of HB = 0

(surf
constant
part change
kro)

$\boxed{\text{eq}^n \text{ of P.O. Asy} = 0}$

P.O.S.L

$\boxed{\Delta = 0}$

Ex. Find centre, pair of asymptotes, equation of conjugate hyperbola for HB $x^2 - 4y^2 - 3xy - 5x + 10y = 0$?

Oblique

P.O. Asymp. $\Rightarrow$ $\boxed{x^2 - 4y^2 - 3xy - 5x + 10y + \lambda = 0}$

$a = 1,\ b = -4,\ c = \lambda,\ h = -\frac{3}{2},\ g = -\frac{5}{2},\ f = 5$

$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

$\boxed{\begin{vmatrix} 1 & -3/2 & -5/2 \\ -3/2 & -4 & 5 \\ -5/2 & 5 & \lambda \end{vmatrix} = 0}$

$1(-4\lambda - 25) + \frac{3}{2}\left(-\frac{3\lambda}{2} + \frac{25}{2}\right) - \frac{5}{2}\left(\ldots\right) = 0$

$\boxed{\lambda = 6}$

POSL or POAs:

$\boxed{x^2 - 4y^2 - 3xy - 5x + 10y + 6 = 0}$

Centre ≡ point of int

$\boxed{x^2 - 4y^2 - 3xy - 5x + 10y + 12 = 0}$

CHB:

$\boxed{HB + CHB = 2\ POAs}$

$0 + \mu = 2(6)$

$\boxed{\mu = 12}$

Ex. Find equation & eccentricity of hyperbola whose equation of asymptotes are $x + y = 3$ & $x - 4y = 2$ and passes through $(5, 0)$.

H.W.

Ex. Find everything for hyperbola : $xy - 3y - 2x = 0$.

H.W.

TODAY's HOMEWORK

MODULE

HYPERBOLA

Exercise – I (TWQ) – Ques: 1 to 18
Exercise – II (LP) – Ques: 1 to 10
Exercise – III (ALMCQ) – Ques: 1,2,3

IV - (PYQ) → Complete

THANK YOU

to all future IITians

PRAYAS 2.0

FOR IIT - JEE 2023

COORDINATE GEOMETRY

HYPERBOLA

LEC – 04

SACHIN JAKHAR

TODAY's GOAL

Rectangular Hyperbola
Properties / Highlights of Hyperbola
OP-QP

LAST CLASS

Asymptotes:

Tangent at '$\underline{\underline{\infty}}$' $\Rightarrow y = \pm \frac{b}{a}x$

Properties of Asymptotes:

① HB & CHB → Same Asy

② (T.A & C.A) are AB's b/w asy

③ eq^n :- $\boxed{H + CHB = 2(P.O.A.)}$

④  $\Rightarrow e = \sec\frac{\theta}{2}$

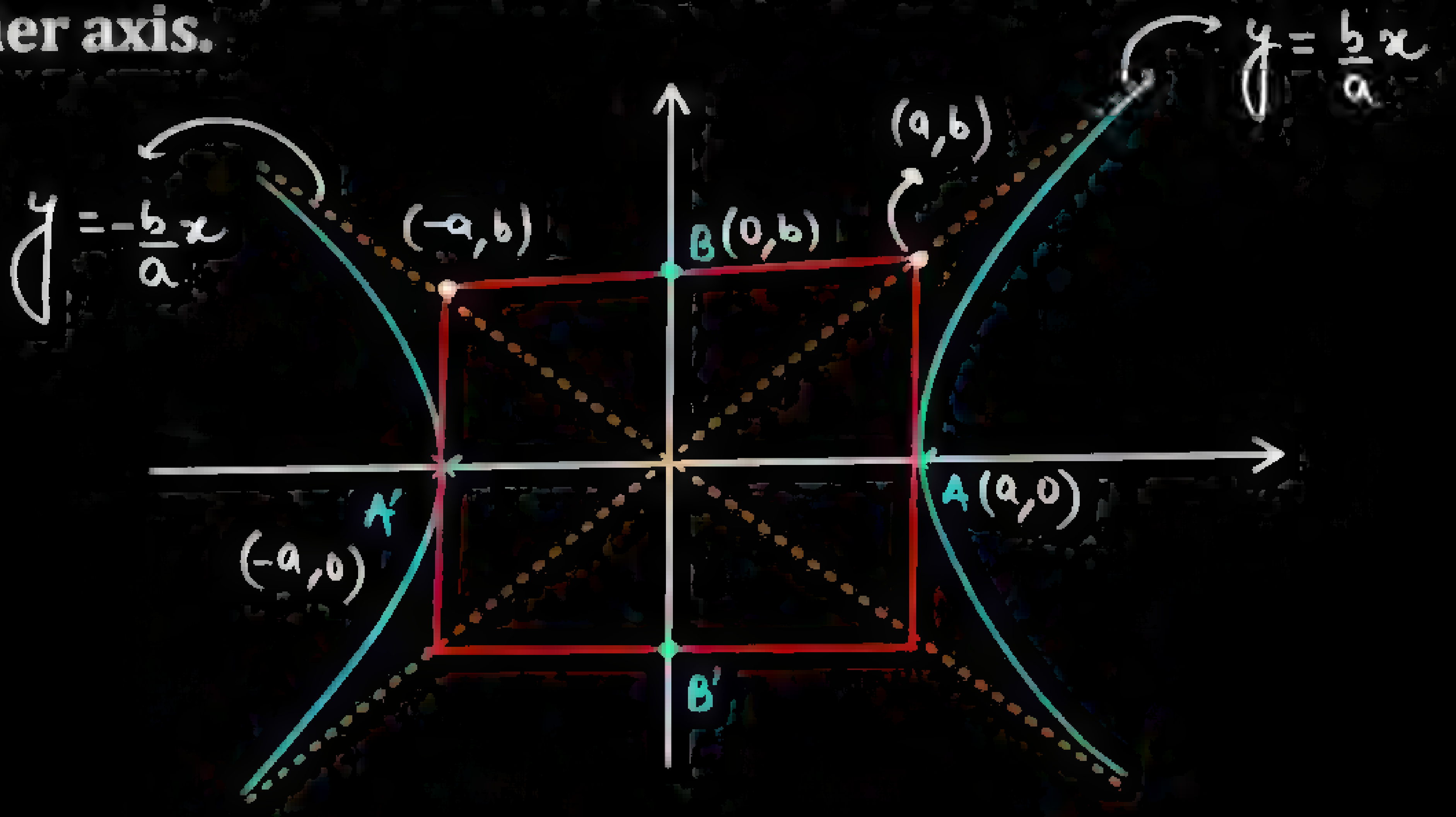
⑤ 3⁴.  $ar(\Delta ABC) = ab$

⑥ 

$p_1p_2 = \frac{a^2b^2}{a^2+b^2}$

Property-07 :

The asymptotes of a hyperbola are the diagonals of he rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.



Remarks :

The point of intersection of tangents at 'θ' and 'ϕ' on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\# \; R\; \boxed{\left(\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right)}$$

Ex. Find equation & eccentricity of hyperbola whose equation of asymptotes are $\boxed{x + y = 3}$ & $\boxed{x - 4y = 2}$ and passes through (5, 0).

Pair of asymptotes:

$$(x+y-3)(x-4y-2)=0$$

$$x^2+xy-3x-4xy-4y^2+12y-2x-2y+6=0$$

$$\boxed{x^2-4y^2-3xy-5x+10y+6=0}$$

eqn of HB: $x^2-4y^2-3xy-5x+10y+\cancel{\lambda}=0$

Pass (5,0)

$$25-25+\lambda=0$$

$$\boxed{\lambda=0}$$

$a=1,\ b=-4,\ h=-\frac{3}{2}$ $\Rightarrow \left(-\frac{3}{2}\right)^2-(1)(-4)$

$$\frac{9}{4}+4=\frac{25}{4}=h^2-ab$$

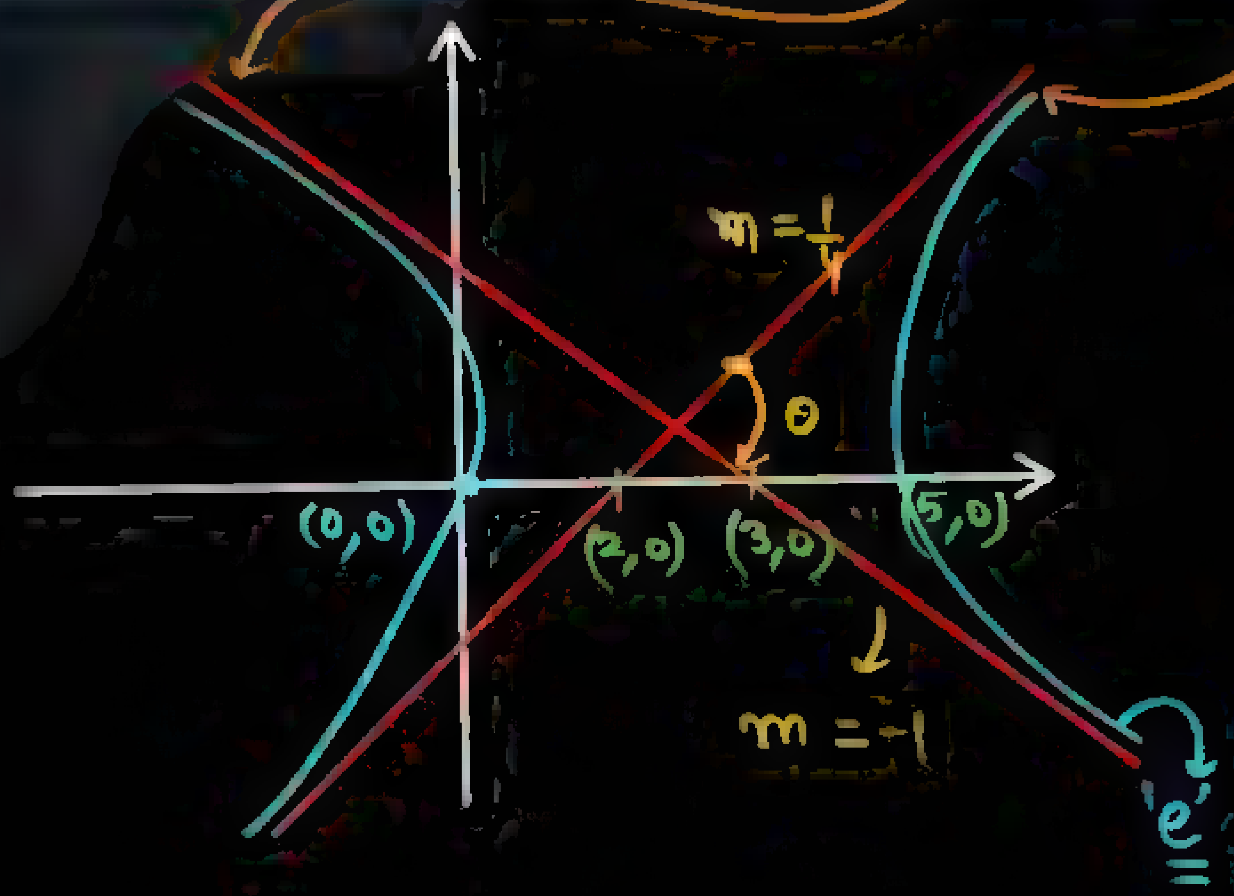
$$\frac{25}{4}=e^2-1$$

$$\frac{25+4}{4}=e^2$$

$$\frac{29}{4}=e^2$$

$$\boxed{e=\frac{\sqrt{29}}{2}}$$

Ex. Find equation & eccentricity of hyperbola whose equation of asymptotes are $\boxed{x + y = 3}$ & $\boxed{x - 4y = 2}$ and passes through (5, 0). ?



$$\# \tan\theta = \frac{\frac{1}{4} - (-1)}{1 + \frac{1}{4}(-1)} = \frac{5/4}{3/4} = \frac{5}{3}$$

$$\cos\theta = \frac{3}{\sqrt{34}}$$

$$2\cos^2\frac{\theta}{2} - 1 = \frac{3}{\sqrt{34}}$$

$$2\cos^2\theta/2 = 1 + \frac{3}{\sqrt{34}} = \frac{\sqrt{34}+3}{\sqrt{34}}$$

$$\cos^2\frac{\theta}{2} = \frac{\sqrt{34}+3}{2\sqrt{34}}$$

$$e = \sec\theta/2$$

$$\# \boxed{e = \sqrt{\frac{2\sqrt{34}}{\sqrt{34}+3}}}$$

Ex. Find everything for hyperbola : $xy - 3y - 2x = 0$.

$x(x-1) - 3(x-1) - 2x$

$\rightarrow$ Oblique HB

$e = \sqrt{2}$

RHB

$a = b$

HB $\xrightarrow{\lambda}$ P.O.A $\longrightarrow \Delta = 0 \longrightarrow \lambda$ ✓

$\boxed{xy - 3y - 2x = 0} \longrightarrow \boxed{xy - 3y - 2x + \lambda = 0}$

$$\Delta = \begin{vmatrix} 0 & \frac{1}{2} & -1 \\ \frac{1}{2} & 0 & -\frac{3}{2} \\ -1 & -\frac{3}{2} & \lambda \end{vmatrix} = 0 \Rightarrow \boxed{\lambda = 6}$$

P.O.A :

$xy - 3y - 2x + 6 = 0$

$y(x-3) - 2(x-3) = 0$

$\boxed{(x-3)(y-2) = 0}$

$\left.\begin{array}{l} x - 3 = 0 \\ y - 2 = 0 \end{array}\right\}$

T.A $\Rightarrow$ $\boxed{y - 2 = x - 3}$

$m = 1$

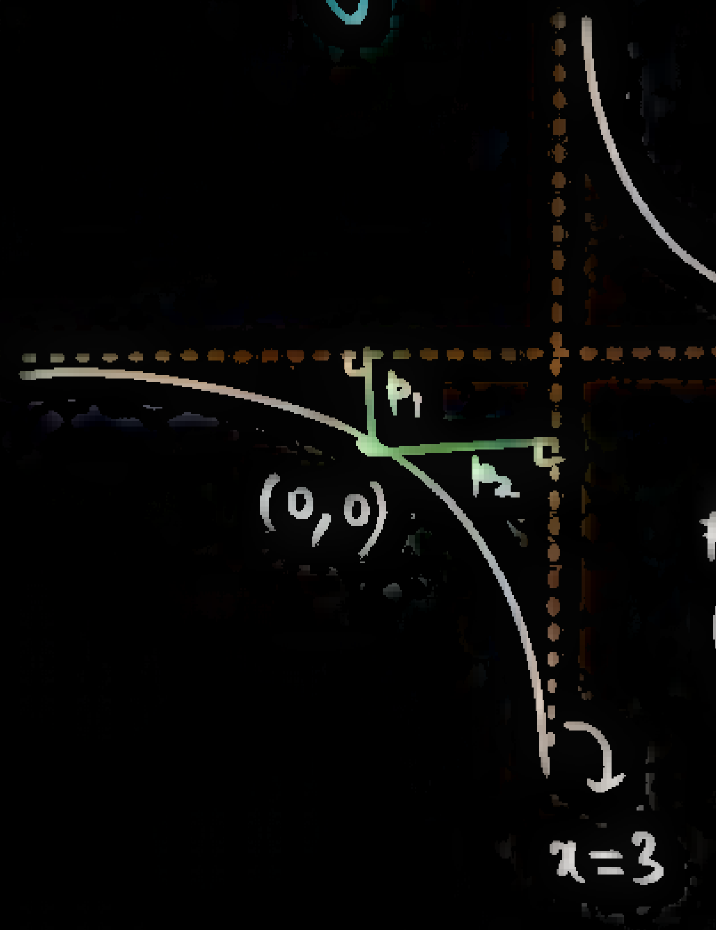
$y = x - 1$

C.A

$m = -1$

$\boxed{y - 2 = -1(x-3)}$

Origin lies on RHB



a = b

\# $p_1 p_2 = \frac{a^2}{2} = \frac{a^2 b^2}{a^2+b^2}$

$2 \times 3 = a^2$

$12 = a^2$

$\boxed{a = \sqrt{12}}$

\# $LR = \frac{2b^2}{a} = 2a$

$\boxed{LR = 2\sqrt{12}}$

RECTANGULAR HYPERBOLA

* also

\# Equilateral HB

The Hyperbola whose:

Length of T.A. = Length of C.A. = Length of L.R.

or

whose eccentricity (e) = $\sqrt{2}$

or

whose asymptotes are perpendicular

or

whose director circle is a point Circle

or

whose 'e' is equal to eccentricity of CHB

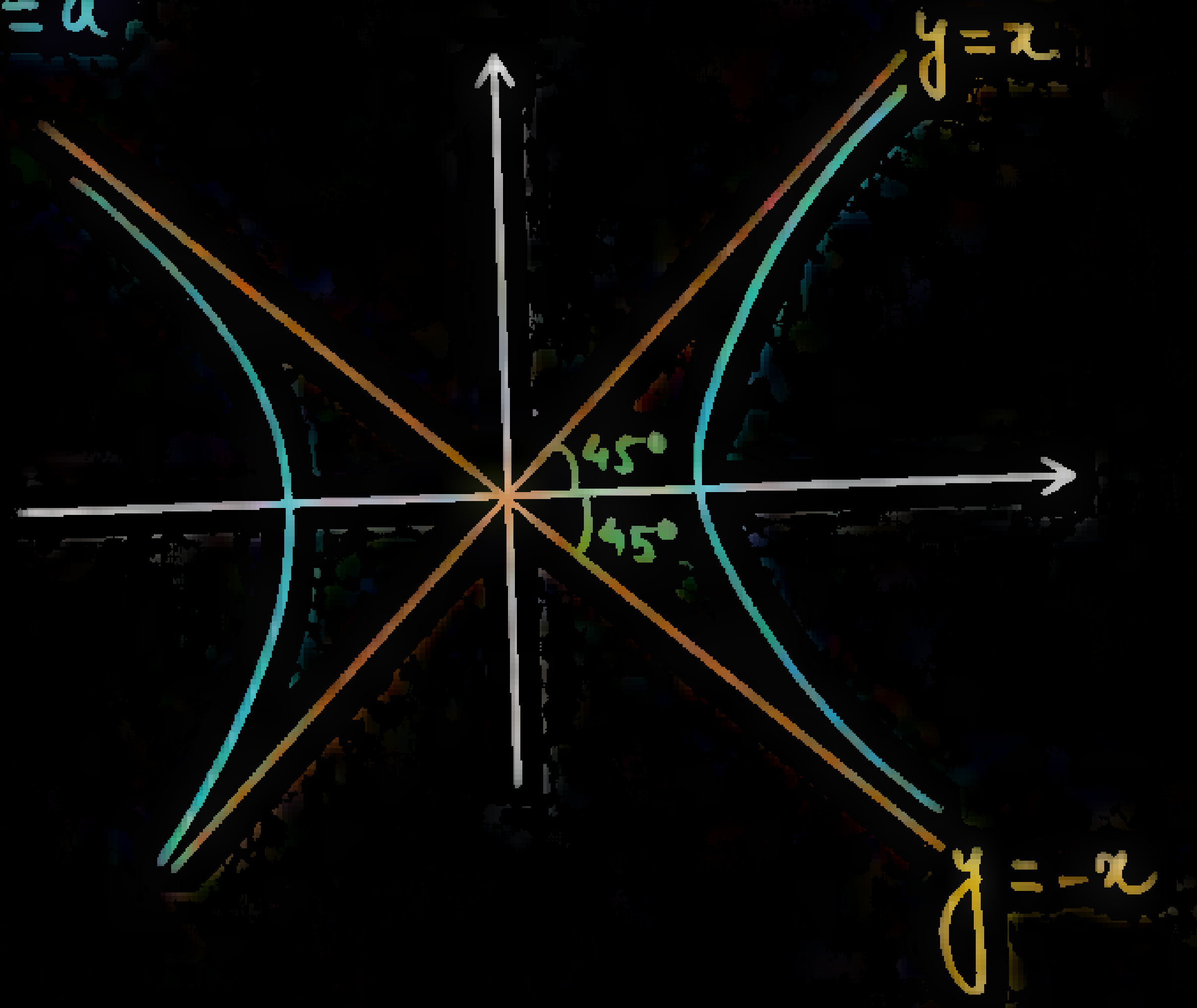
or

whose equation is : $x^2 - y^2 = a^2$

\# all results are valid just put $(a = b)$

Asymptotes :- $\boxed{y = \pm x}$

$x^2 - y^2 = a^2$



STANDARD RECTANGULAR HYPERBOLA

 # For which asymptotes are co-ordinate axes.

X-axis $\Rightarrow$ $y = 0$
Y-axis $\Rightarrow$ $x = 0$

} Asymp

Pair of asymp

$\boxed{xy = 0}$

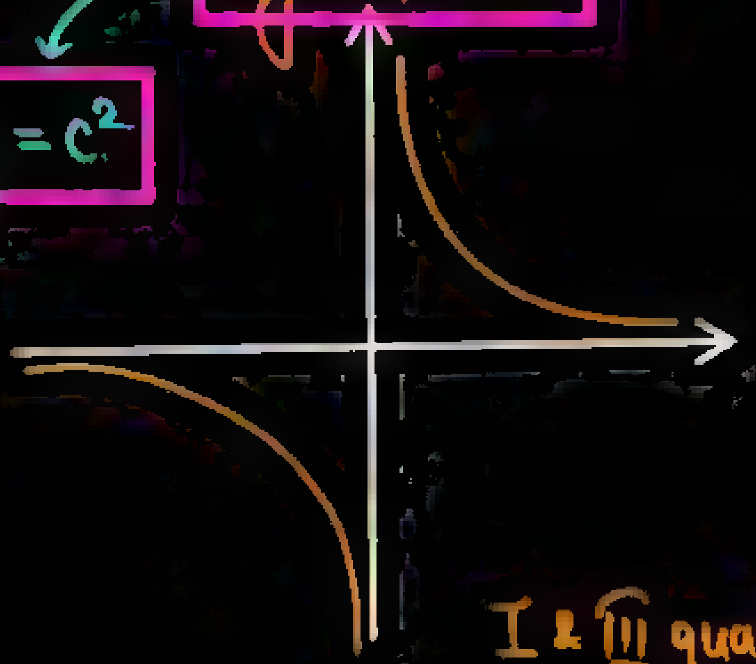
eq^n of HB :-

$xy = c$

$\boxed{xy = c}$

$\boxed{xy = c,\ c > 0}$

$\boxed{xy = c^2}$



I & III quad

$\boxed{xy = c;\ c < 0}$ OR

$\boxed{xy = -c^2}$

II & IV quad

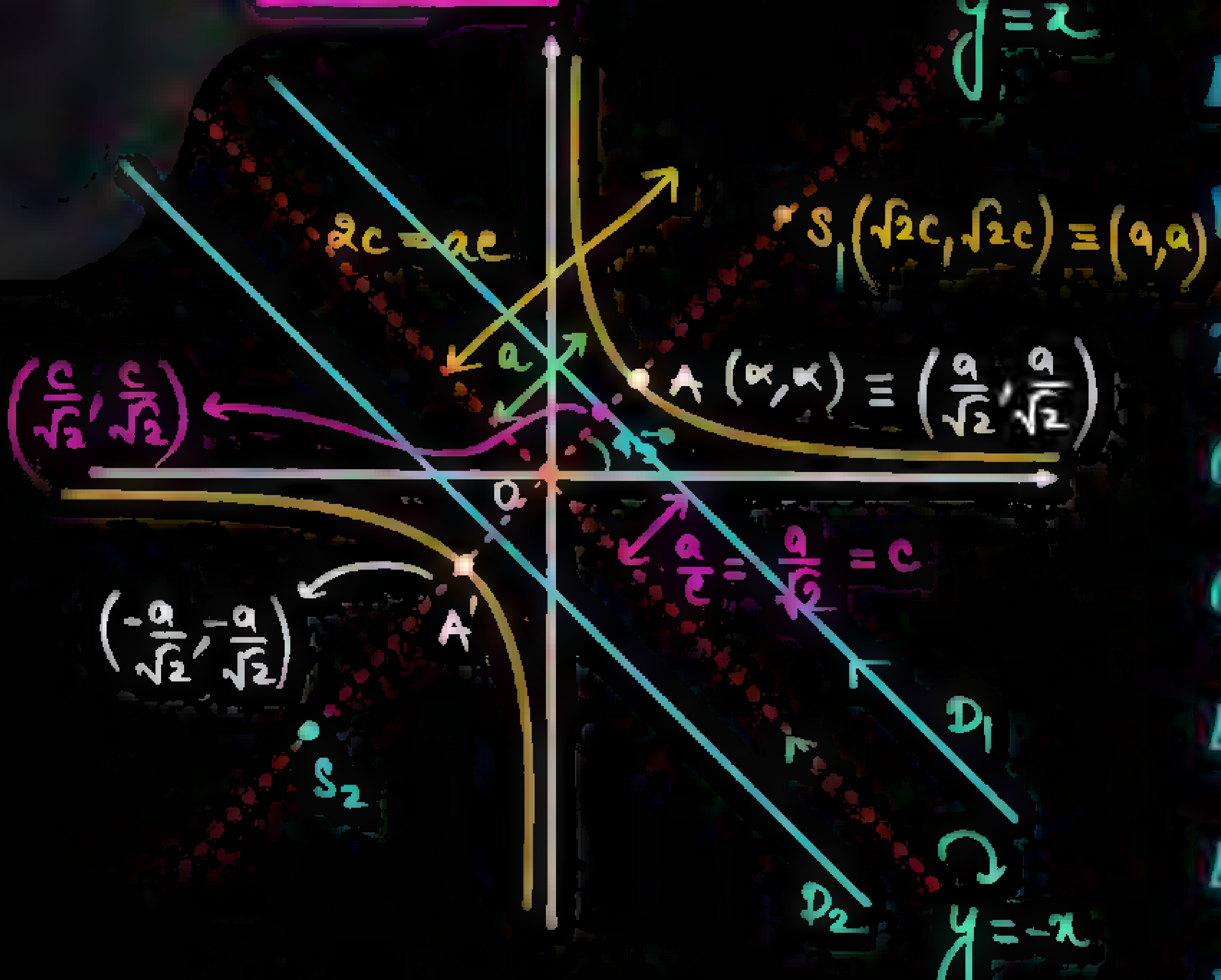
$\boxed{x^2 - y^2 = a^2}$

for standard HB

Rotation of axis by $45°$

ALL TOGETHER

$\boxed{xy = c^2}$ $(a, c > 0)$



$OA = a = \sqrt{\alpha^2 + \alpha^2} \Rightarrow a = \sqrt{2}\alpha$

$\boxed{a = \text{semi T.A. or semi C.A.}}$

$\boxed{c^2 = \text{given const in SRHB}}$

Relation b/w 'c' & 'a': $\boxed{a = \sqrt{2}c}$

Foci: $S_1(a, a) \equiv (\sqrt{2}c, \sqrt{2}c)$ & $S_2(-\sqrt{2}c, -\sqrt{2}c) \equiv (-a, -a)$

Vertices: $\boxed{A\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right) \equiv (c, c)}$ & $\boxed{A'\left(\frac{-a}{\sqrt{2}}, \frac{-a}{\sqrt{2}}\right) \equiv (-c, -c)}$

Transverse Axis: $y = x$

Conjugate Axis: $y = -x$

Centre: Origin

Directrix$_1$: $y = -x + \sqrt{2}c$

Directrix$_2$: $y = -x - \sqrt{2}c$

Parametric Eqn: $\boxed{\left(ct, \frac{c}{t}\right)}$ $t \in R - \{0\}$ Parameter

RHB :

$\boxed{xy = c^2}$

CHB

$\boxed{xy = -c^2}$

Point $\equiv \left(ct, -\frac{c}{t}\right)$

TANGENT & NORMAL

$\boxed{xy = c^2}$

Tangent:

(i) At $P(x_1, y_1)$: $\dfrac{xy_1 + yx_1}{2} = c^2 \Rightarrow \boxed{xy_1 + yx_1 = 2c^2}$ $\to m_T = \dfrac{-y_1}{x_1}$

(ii) Parametric Form:

$P(x_1, y_1) \equiv \left(ct, \dfrac{c}{t}\right)$

$x\left(\dfrac{c}{t}\right) + y\,ct = 2c^2$

$\boxed{\dfrac{x}{t} + yt = 2c}$ $\to m_T = -\dfrac{1}{t^2} < 0$

Normal:

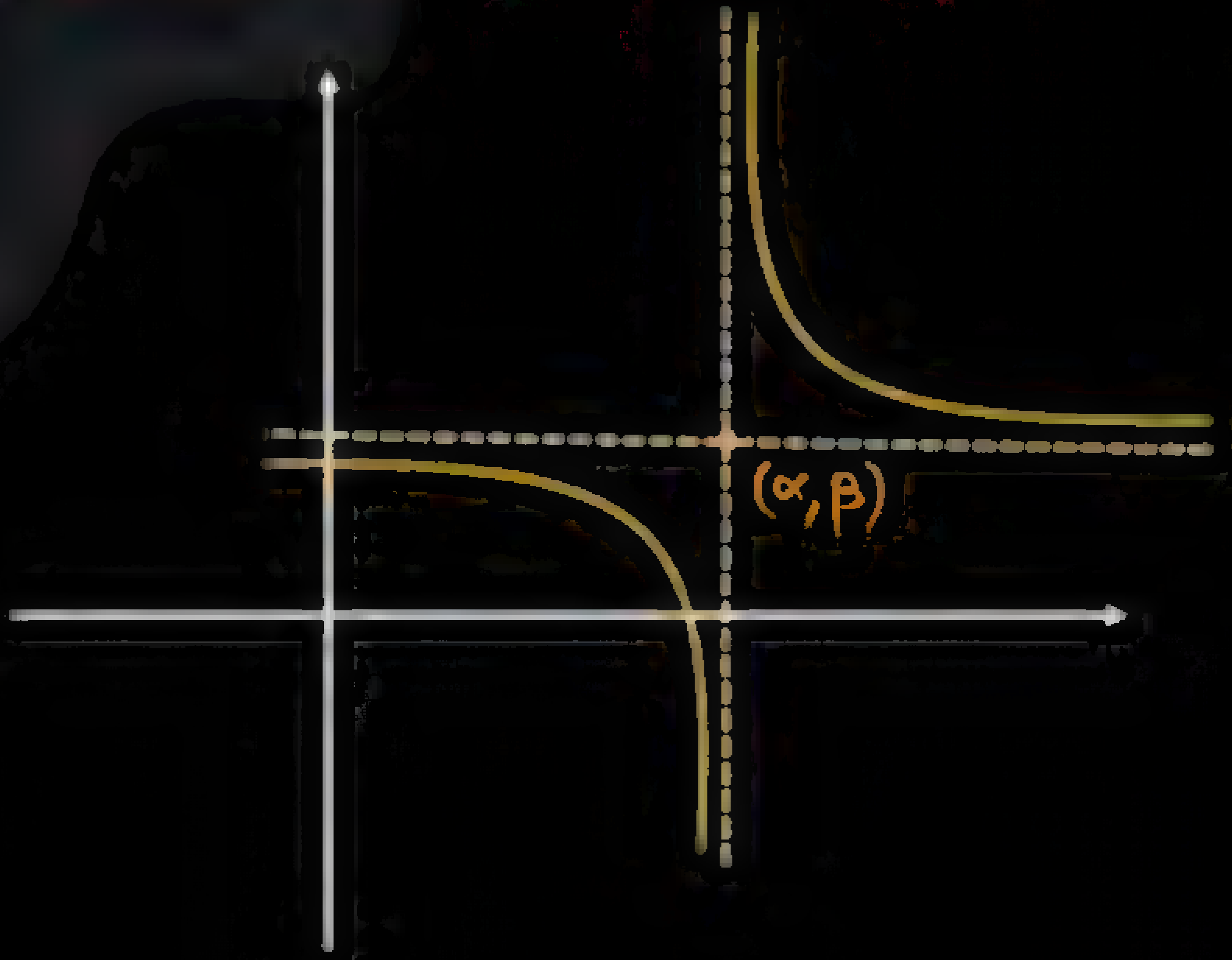
(i) At $P(x_1, y_1)$: $y - y_1 = m_N(x - x_1) \Rightarrow \boxed{y - y_1 = \dfrac{x_1}{y_1}(x - x_1)}$

(ii) Parametric Form:

$P(x_1, y_1) \equiv \left(ct, \dfrac{c}{t}\right)$

$y - \dfrac{c}{t} = \dfrac{ct}{\left(\frac{c}{t}\right)}(x - ct) \Rightarrow \boxed{y - \dfrac{c}{t} = t^2(x - ct)}$

Shifted Standard RHB:



SRHB:

$\boxed{xy = c^2}$

Centre (α, β)

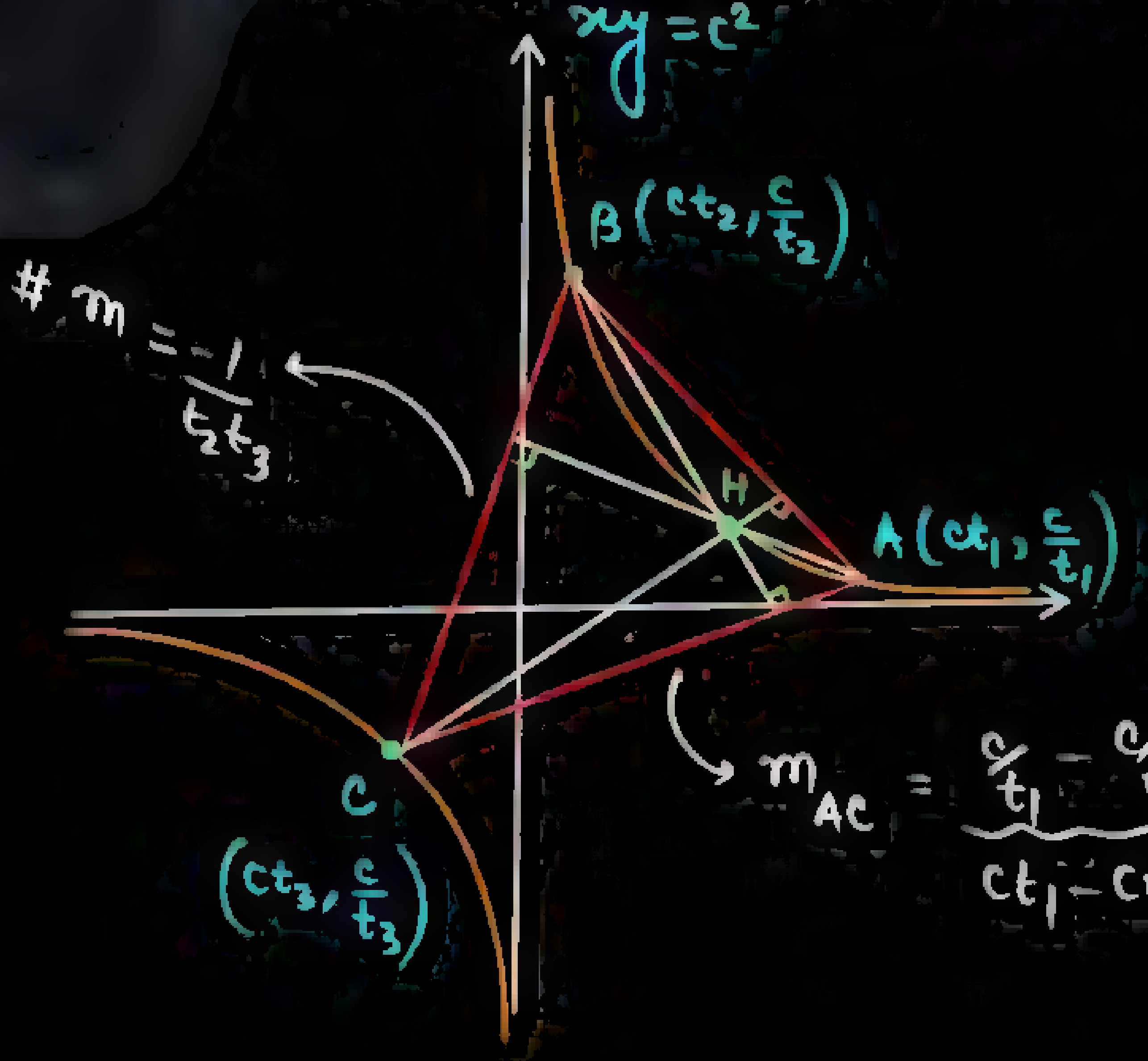
$x \to x - \alpha$

$y \to y - \beta$

$$\boxed{(x-\alpha)(y-\beta) = c^2}$$

Note:

Show that the orthocenter of the triangle formed by 3 points lying on a rectangular Hyperbola always lies on the same Rectangular Hyperbola.



$$m_{AC} = \frac{\frac{c}{t_1} - \frac{c}{t_3}}{ct_1 - ct_3} = \frac{c(t_3 - t_1)}{t_1t_3\,c(t_1 - t_3)} = \frac{-1}{t_1t_3}$$

eqn of BH : $y - \frac{c}{t_2} = t_1t_3(x - ct_2)$

Similarly eqn of AH : $y - \frac{c}{t_1} = t_2t_3(x - ct_1)$

P.O.I.

$$\frac{c}{t_1} - \frac{c}{t_2} = t_1t_3x - t_2t_3x + \cancel{ct_1t_2t_3} - \cancel{ct_1t_2t_3}$$

$$c\left(\frac{t_2 - t_1}{t_1t_2}\right) = t_3x(t_1 - t_2)$$

$\boxed{\frac{-c}{t_1t_2t_3} = x}$, $\boxed{y = -ct_1t_2t_3}$

$xy = c^2$

Sachin Sir Summary :

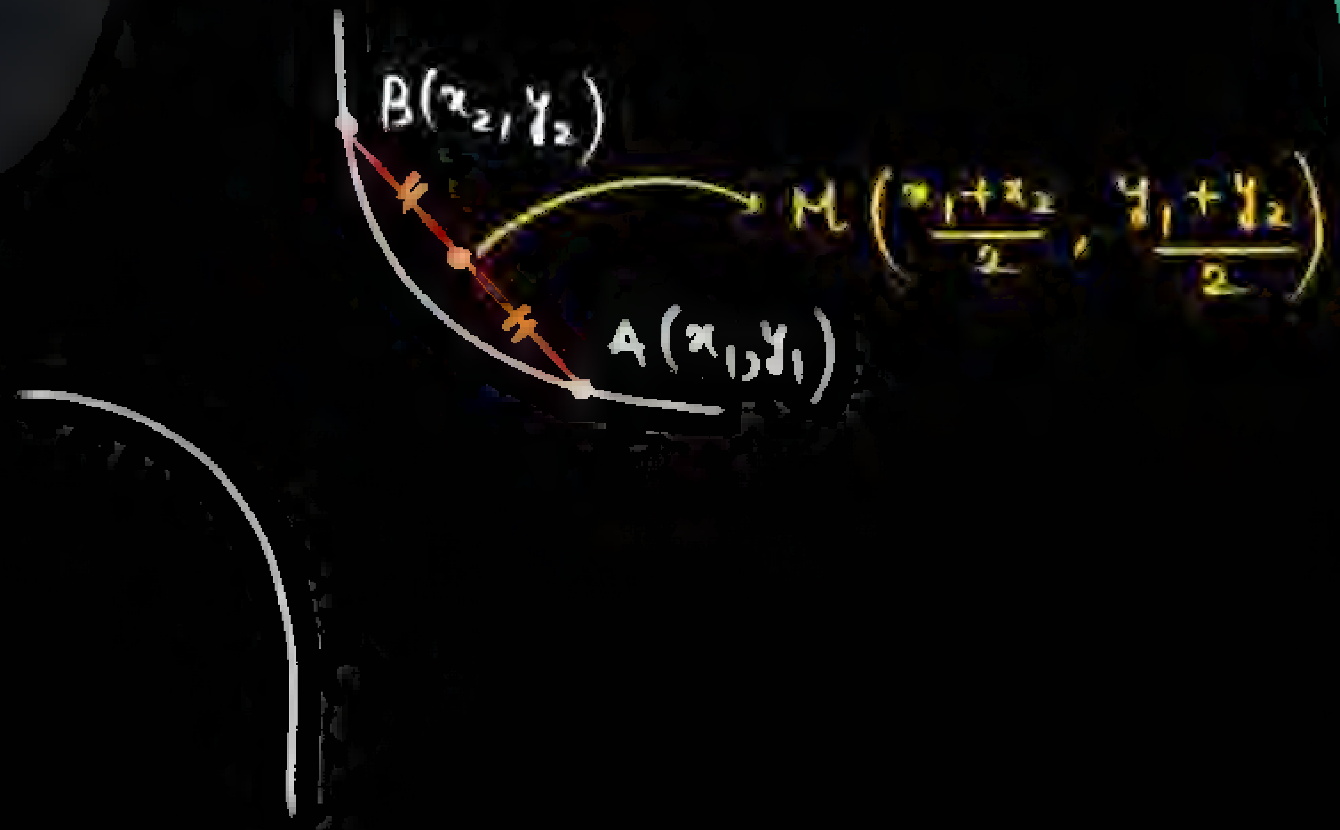
R.H.B ⇒ $\boxed{a=b}$ → $\boxed{x^2-y^2=a^2}$ → $\boxed{e=\sqrt{2}}$

Asymp. ⊥r

SRHB : "x & y axis are asymptotes"

→ $\boxed{xy=c^2}$ → $\boxed{a=\sqrt{2}c}$

Ex. Show that equation of chord joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ on R.H.B $\boxed{xy = c^2}$ is $\dfrac{x}{x_1+x_2} + \dfrac{y}{y_1+y_2} = 1$



$xy - c^2 = 0$

eq^n of AB with m.p. M :

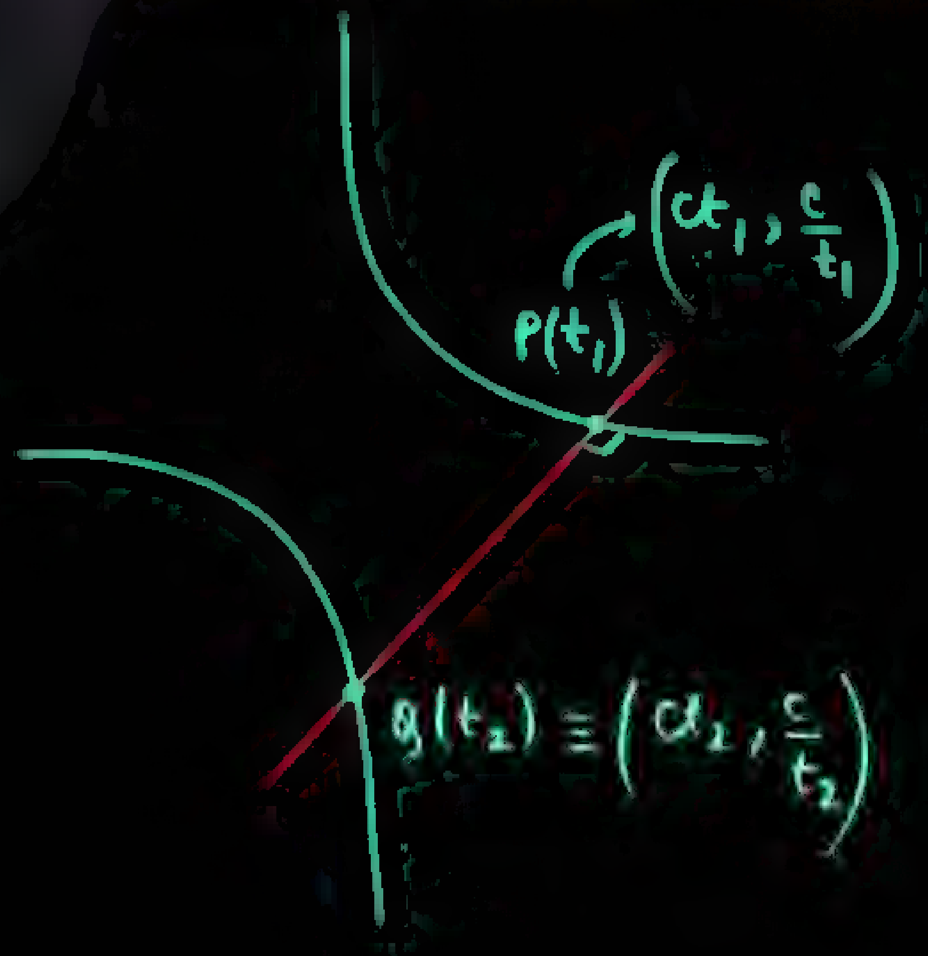
$T_1 = S_1$

$$\frac{x\left(\frac{y_1+y_2}{2}\right) + y\left(\frac{x_1+x_2}{2}\right)}{2} - \cancel{c^2} = \left(\frac{x_2+x_1}{2}\right)\left(\frac{y_2+y_1}{2}\right) - \cancel{c^2}$$

$$\frac{(y_1+y_2)x + y(x_1+x_2)}{\cancel{4}} = \frac{(x_2+x_1)(y_2+y_1)}{\cancel{4}}$$

$\div$

Ex. If normal drawn at point $P(t_1)$ to hyperbola $xy = c^2$ meets it again at $Q(t_2)$ then value of $t_1^3 t_2 =$



Normal at $P(t_1)$: $\boxed{y - \frac{c}{t_1} = t_1^2 (x - ct_1)}$

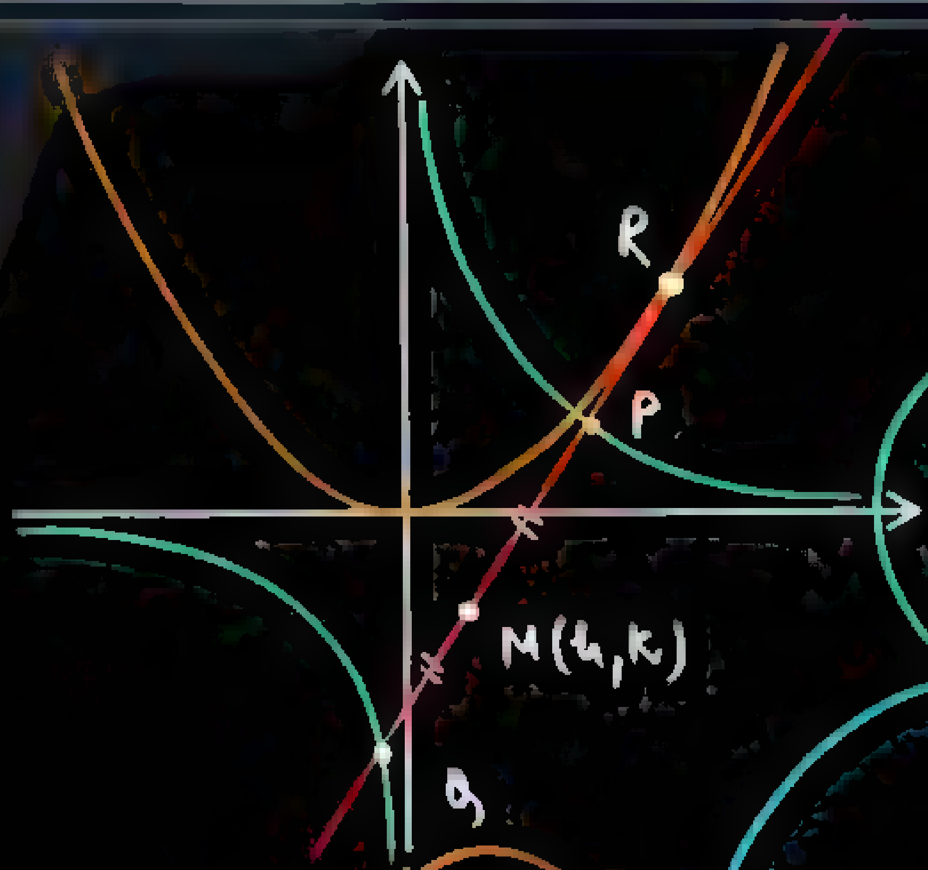
$Q\left(ct_2, \frac{c}{t_2}\right)$ pass

$$\frac{c}{t_2} - \frac{c}{t_1} = t_1^2 (ct_2 - ct_1)$$

$$\frac{-1\,\cancel{(ct_1 - ct_2)}}{t_1 t_2} = t_1^2 \,\cancel{(ct_2 - ct_1)}$$

$$\boxed{-1 = t_1^3 t_2}$$

Q. A variable tangent to $x^2 = 4ay$ intersects $xy = c^2$ in P and Q. Find the locus of mid-point of PQ.



Any tangent : $\boxed{y = mx - am^2}$

eqⁿ of PQ with m.p. M(h,k) : $T_1 = S_1$

$$\frac{xk + yh}{2} - \cancel{c^2} = hk - \cancel{c^2}$$

$$\boxed{xk + yh = 2hk}$$

$$\boxed{\frac{x}{2h} + \frac{y}{2k} = 1}$$

"Same" Compare

$$mx - y = am^2$$

$$\boxed{\frac{x}{am} - \frac{y}{am^2} = 1}$$

$$\frac{1}{2h} = \frac{1}{am} \quad \& \quad -\frac{1}{am^2} = \frac{1}{2k}$$

$$m = \frac{2h}{a} \qquad -\frac{2k}{a} = m^2$$

$$-\frac{2k}{\cancel{a}} = \frac{4h^2}{a^2} \Rightarrow -ak = 2h^2$$

locus

$$-\frac{a}{2}y = x^2$$

Q.

Show that the mid points of focal chords of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lies on another similar hyperbola.

having same eccentricity

H.W

Q. Any tangent to rectangular hyperbola $x^2 - y^2 = 9$ intersects parabola $y^2 = 8x$ at A & B. If point of intersection of tangents at A & B lies on an ellipse whose eccentricity is ______.

#HW

Q.

A common tangent T to the curves $C_1 : \frac{x^2}{4} + \frac{y^2}{9} = 1$ and $C_2 : \frac{x^2}{42} - \frac{y^2}{143} = 1$ does not pass through the fourth quadrant. If T touches C_1 at (x_1, y_1) and C_2 at (x_2, y_2), then $|2x_1 + x_2|$ is equal to ______.

[JEE Mains-2022]

H.W.

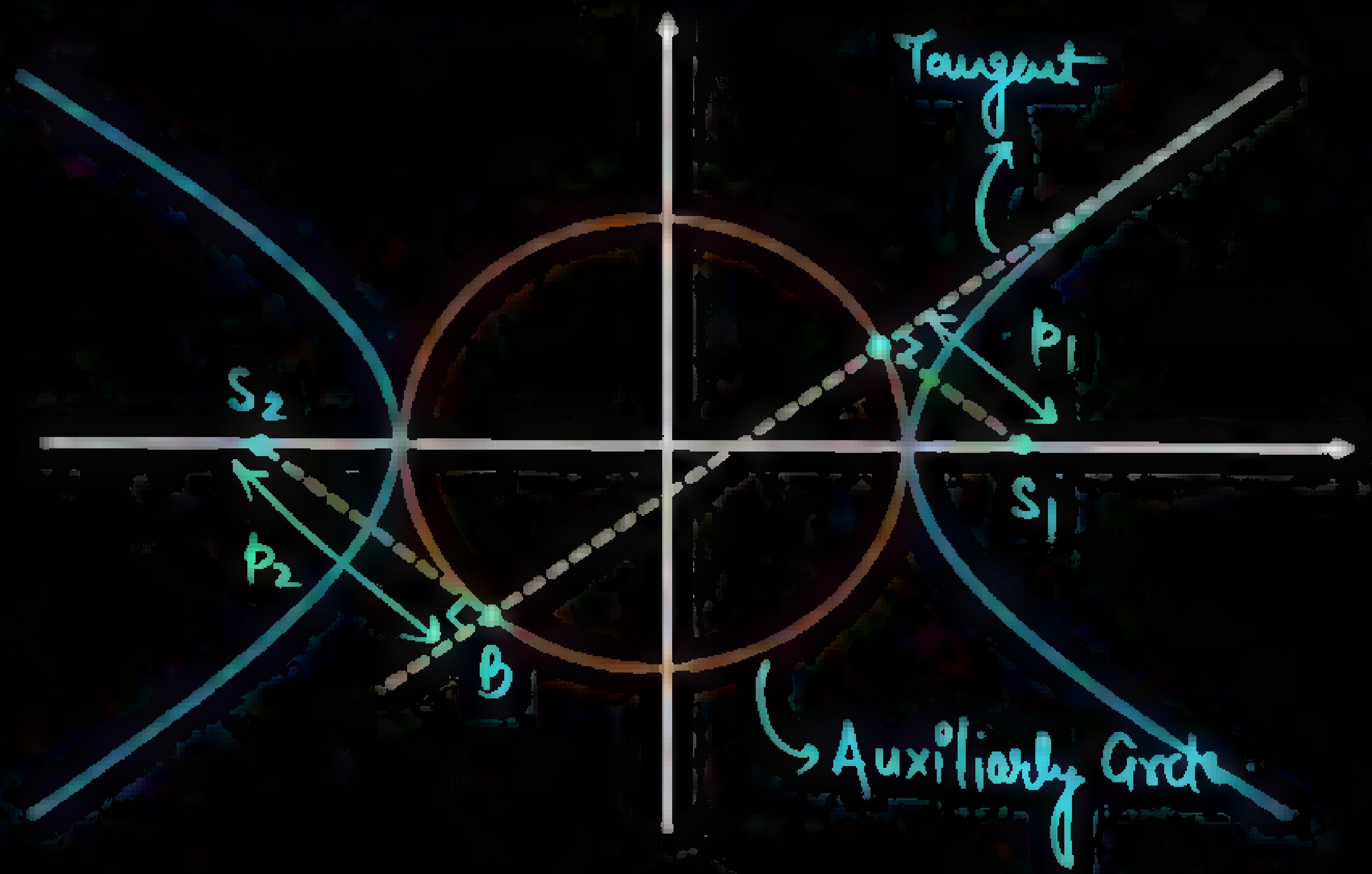
Diameter:

eq^n : $\boxed{y^2 = \frac{b^2}{a^2 m} x}$

PROPERTIES OF HB

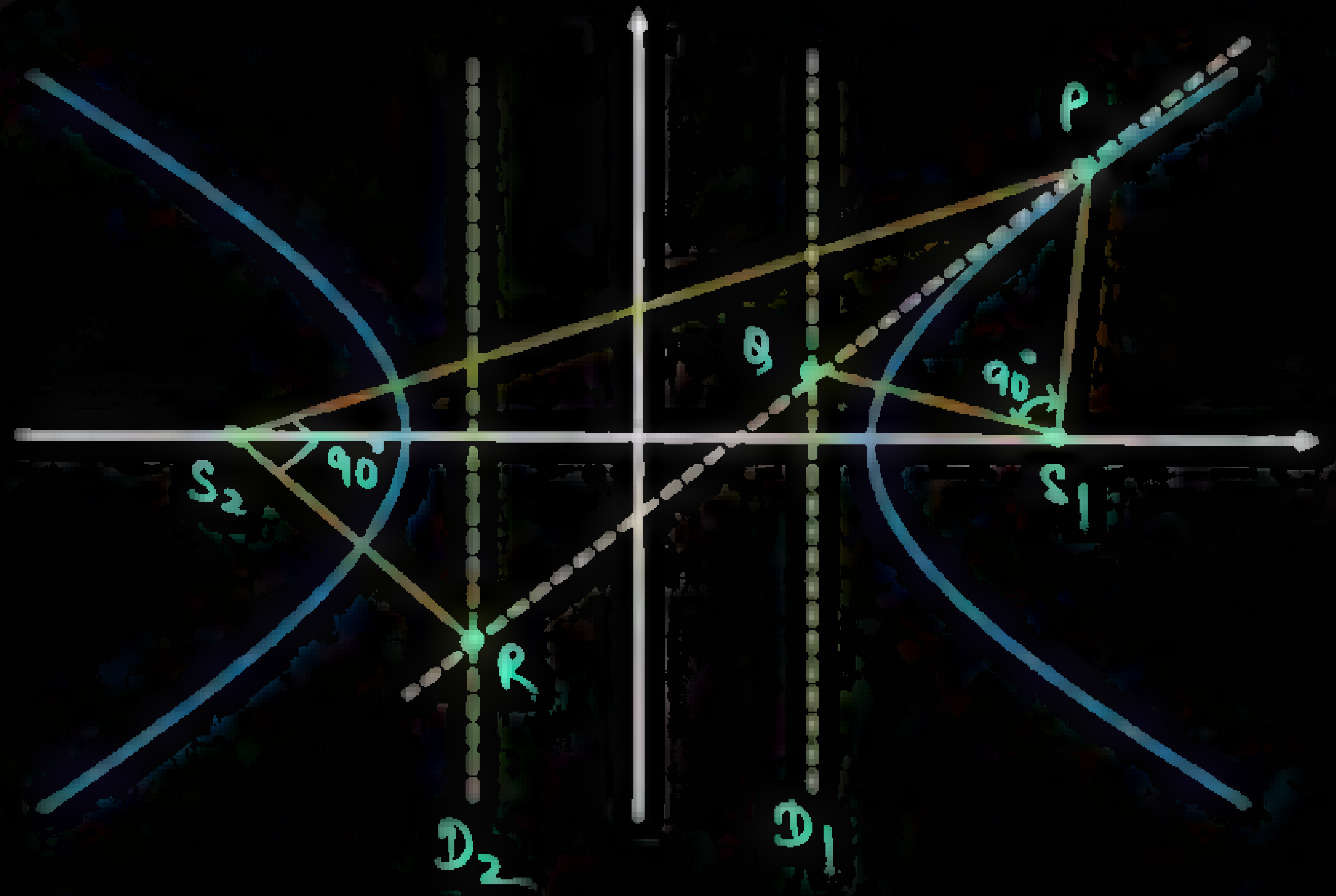
P-1: Locus of foot of perpendicular drawn from foci on any tangent is Auxiliary Circle.

P-2: Product of lengths of perpendiculars from foci on Tangent is always constant & equals to (semi-conjugate axis)^2

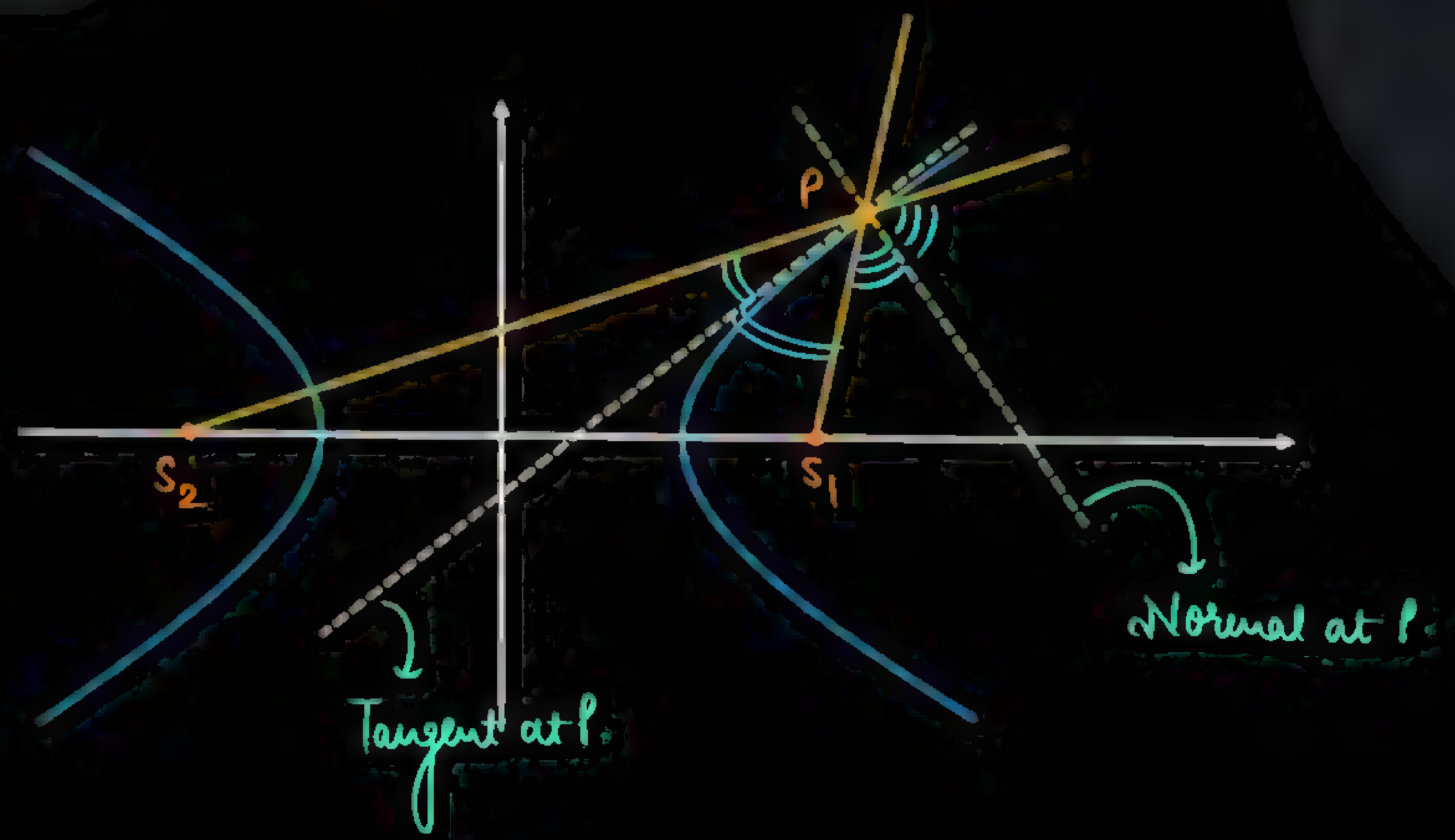


$$\# \quad \boxed{p_1 p_2 = (\text{semi CA})^2 = b^2}$$

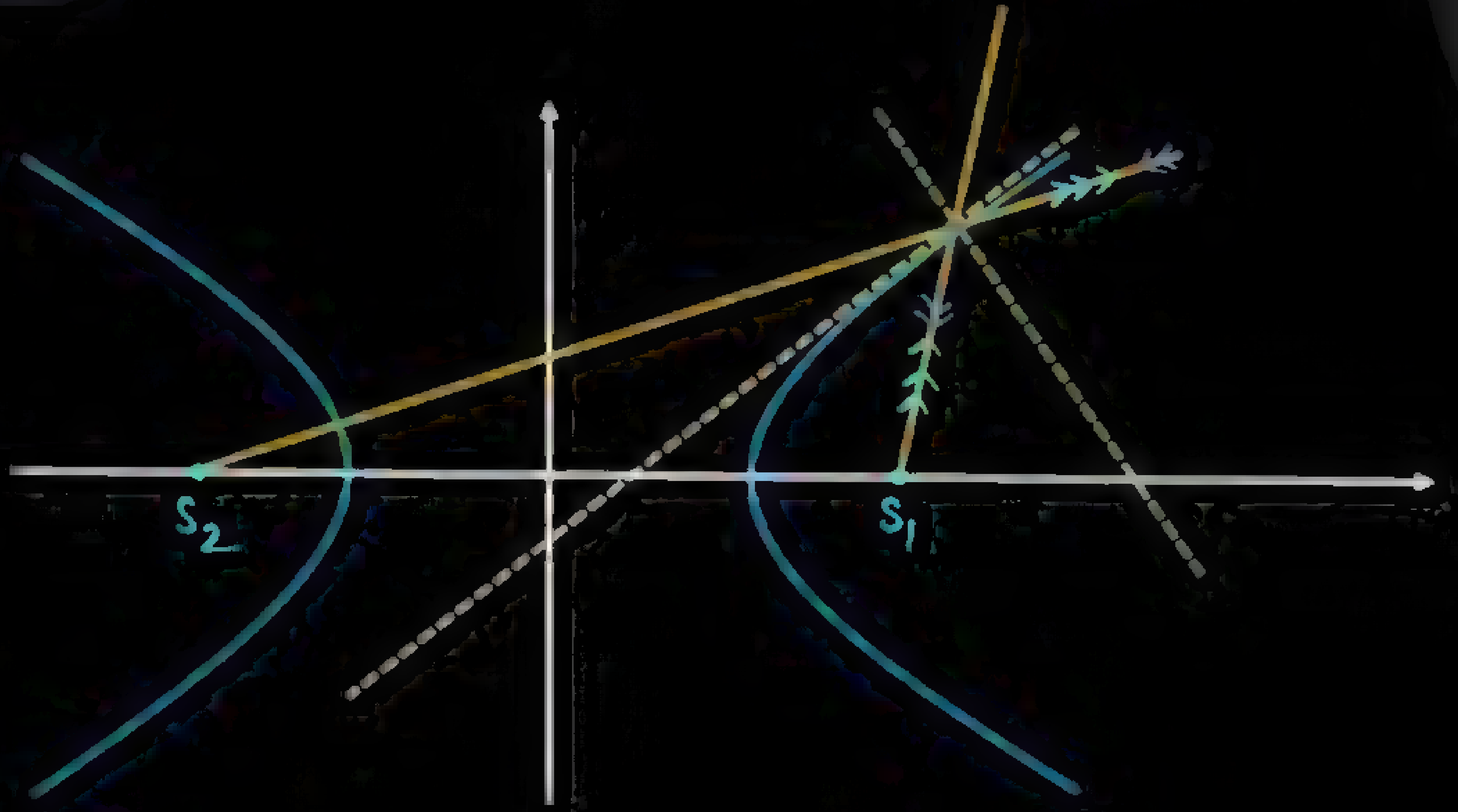
P-3: Portion of tangent intercepted between point of contact and directrix subtend 90° at corresponding focus.



P-4: Tangent and Normal at any point P bisects the angle between focal distances (PS_1 & PS_2).



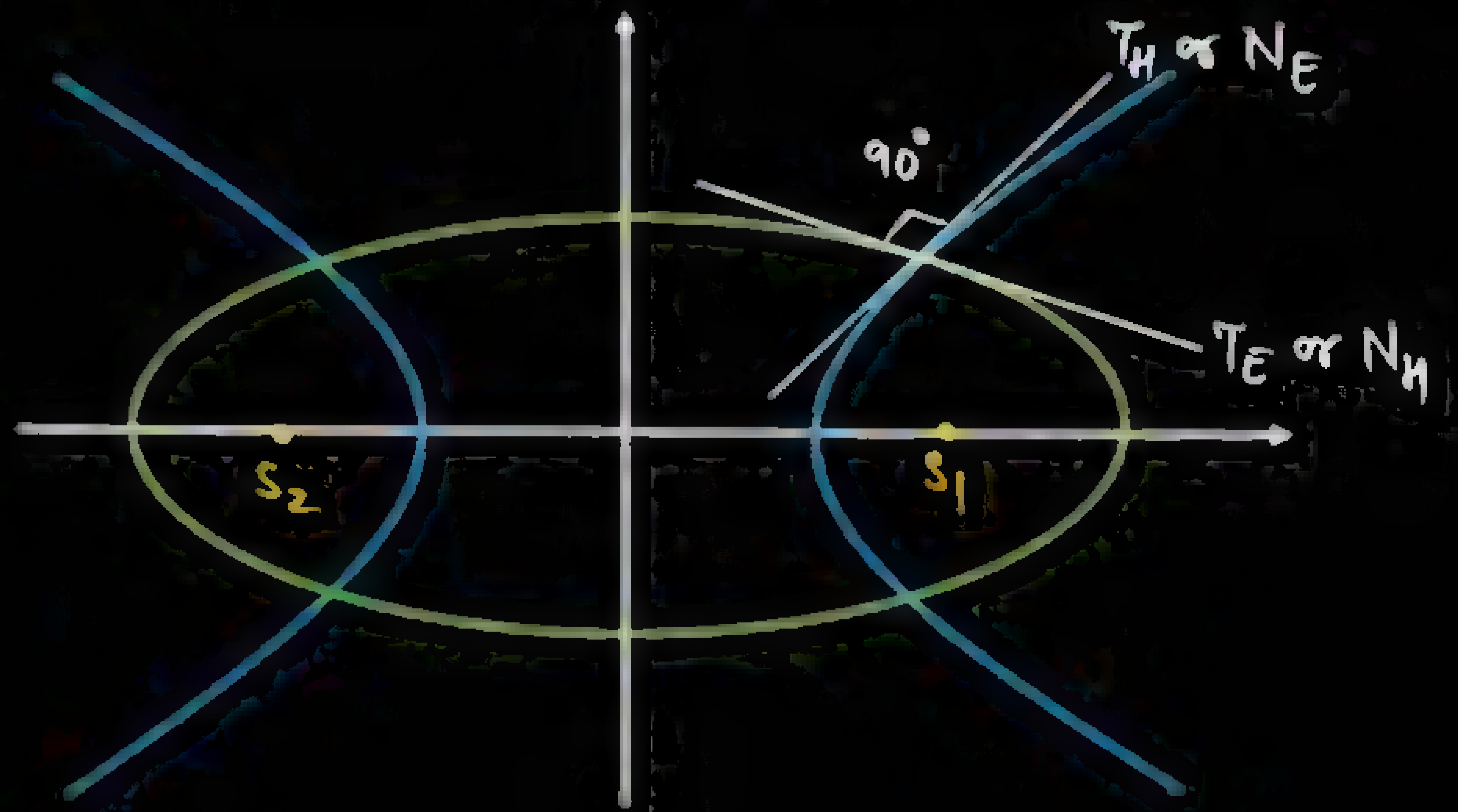
REFLECTION PROPERTY: *Any ray passing through one focus, after reflection from Hyperbola it passes from another focus.*



P-5: Using Reflection Property we can say that:

If Ellipse & Hyperbola are confocal (having same foci) then they are Orthogonal (angle between tangents at point of intersection is 90^0)

Conversely if Ellipse & Hyperbola are $\boxed{\text{Orthogonal}}$ they are $\boxed{\text{Confocal.}}$



Q. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then

[IIT-JEE-2009 (Paper-2)]

A ✓ Equation of ellipse is $x^2 + 2y^2 = 2$

B ✓ The foci of ellipse are $(\pm1, 0)$

C ✗ Equation of ellipse is $x^2 + 2y^2 = 4$

$x^2+2y^2=2 \Leftarrow \frac{x^2}{2}+\frac{y^2}{1}=1$

D ✗ The foci of ellipse are $(\pm\sqrt{2}, 0)$

$e_H = \frac{1}{e_E}$

$\sqrt{2} = \frac{1}{e_E}$

$\boxed{e_E = \frac{1}{\sqrt{2}}}$

$\boxed{\frac{x^2}{\left(\frac{1}{2}\right)} - \frac{y^2}{\left(\frac{1}{2}\right)} = 1}$

$e_H = \sqrt{2}$

$\text{foci} \equiv (\pm ae, 0)$

$\equiv \left(\pm \frac{1}{\sqrt{2}}(\sqrt{2}), 0\right)$

$\equiv (\pm 1, 0)$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2 = 2$

$ae = 1$

$a = \frac{1}{e} = \sqrt{2}$

$\frac{1}{2} = 1 - \frac{b^2}{2} \Leftarrow e_E^2 = 1 - \frac{b^2}{a^2}$

$\frac{b^2}{2} = \frac{1}{2} \Rightarrow b^2 = 1$

Q. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

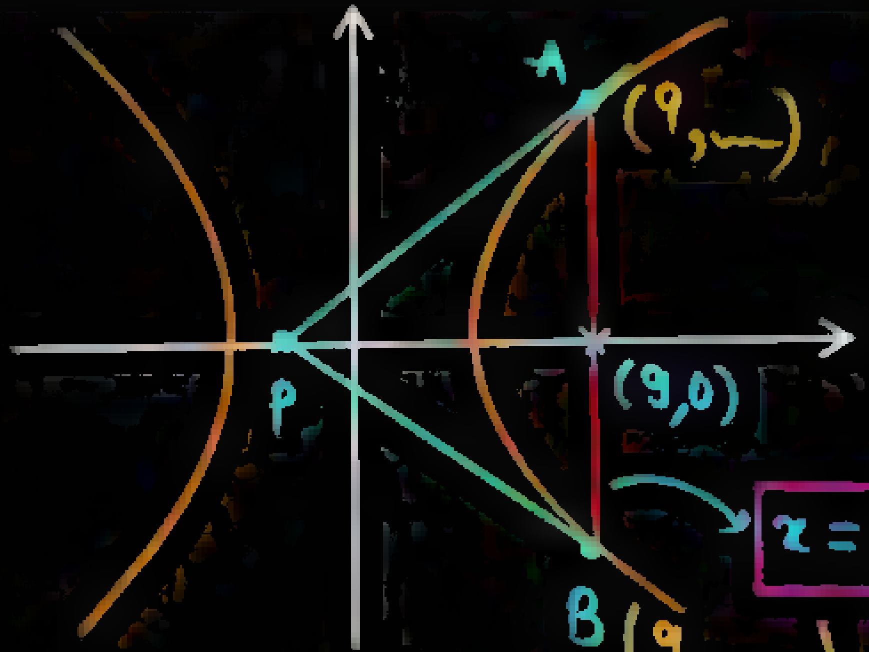
[JEE-1999, 2M]

(A) $9x^2 - 8y^2 + 18x - 9 = 0$

(B) $9x^2 - 8y^2 - 18x + 9 = 0$ ✓

(C) $9x^2 - 8y^2 - 18x - 9 = 0$

(D) $9x^2 - 8y^2 + 18x + 9 = 0$



$\boxed{x = 9}$

$\boxed{PA \cdot PB = 0}$

Method I :
find A & B
Tangent at A & B
Pair ✓

$x^2 - y^2 - 9 = 0$

$\boxed{T_1^2 = SS_1}$

$(x(1) - y(0) - 9)^2 = (x^2 - y^2 - 9)(1 - 0 - 9)$

$(x-9)^2 = -8x^2 + 8y^2 + 72$

$x^2 + 81 - 18x$

Method-II : $P(\alpha, \beta)$

COC $\Rightarrow T_1 = 0$

$\boxed{x\alpha - y\beta = 9}$

comp

$\left.\begin{matrix}\alpha = 1 \\ \beta = 0\end{matrix}\right\}$

$P(1, 0)$

Q. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then

[JEE-1998, 2M]

A $x_1 + x_2 + x_3 + x_4 = 0$

B $y_1 + y_2 + y_3 + y_4 = 0$

C $x_1x_2x_3x_4 = c^4$

D $y_1y_2y_3y_4 = c^4$

$x^2 + y^2 = a^2$

$xy = c^2 \Rightarrow y = \frac{c^2}{x}$

$x^2 + \left(\frac{c^2}{x}\right)^2 = a^2$

$x^2 + \frac{c^4}{x^2} = a^2$

$x^4 + c^4 = a^2x^2$

$\boxed{x^4 + 0x^3 - a^2x^2 + 0x + c^4 = 0}$ → x_1, x_2, x_3, x_4

Product $= c^4$

Sum of Roots $= 0$

General Circle :-

$$\boxed{x^2+y^2+2gx+2fy+d=0}$$

&

SRHB :-

$$\boxed{xy=c^2}$$

any point

$\left(ct, \frac{c}{t}\right)$

$$(ct)^2+\left(\frac{c}{t}\right)^2+2g(ct)+2f\left(\frac{c}{t}\right)+d=0$$

$$c^2t^2+\frac{c^2}{t^2}+2gct+\frac{2fc}{t}+d=0$$

$$c^2t^4+c^2+2gct^3+2fct+dt^2=0$$

$$c^2t^4+(2gc)t^3+dt^2+(2fc)t+c^2=0$$

Product of roots $=\frac{c^2}{c^2}=1=t_1t_2t_3t_4$

Point of Int

Roots

$P(t_1)$

$Q(t_2)$

$R(t_3)$

$S(t_4)$

Q. Consider the hyperbola $\frac{x^2}{100} - \frac{y^2}{64} = 1$ with foci at S and S_1, where S lies on the positive x-axis. Let P be a point on the hyperbola, in the first quadrant. Let $\angle SPS_1 = \alpha$, with $\alpha < \frac{\pi}{2}$. The straight line passing through the points S and having the same slope as that of the tangent at P to the hyperbola, intersects the straight line S_1P at P_1. Let δ be the distance of P from the straight line SP_1, and $\beta = S_1P$. Then the greatest integer less than or equal to $\frac{\beta\delta}{9}\sin\frac{\alpha}{2}$ is ______.

[JEE (Adv.)-2022]

$a = 10,\ b = 8$

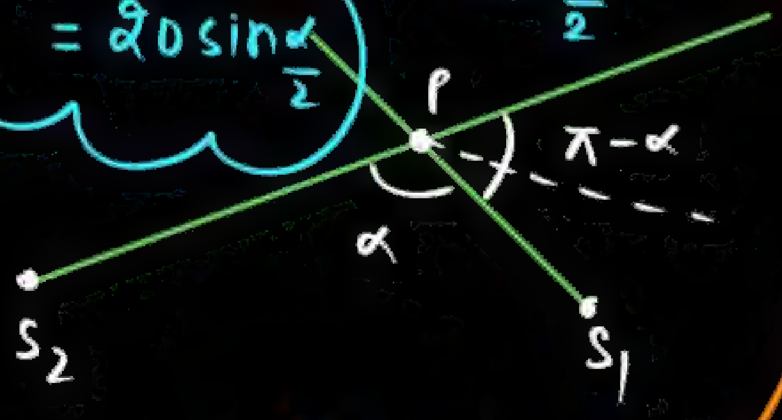
$e^2 = 1 + \frac{64}{100}$

$= \frac{164}{100}$

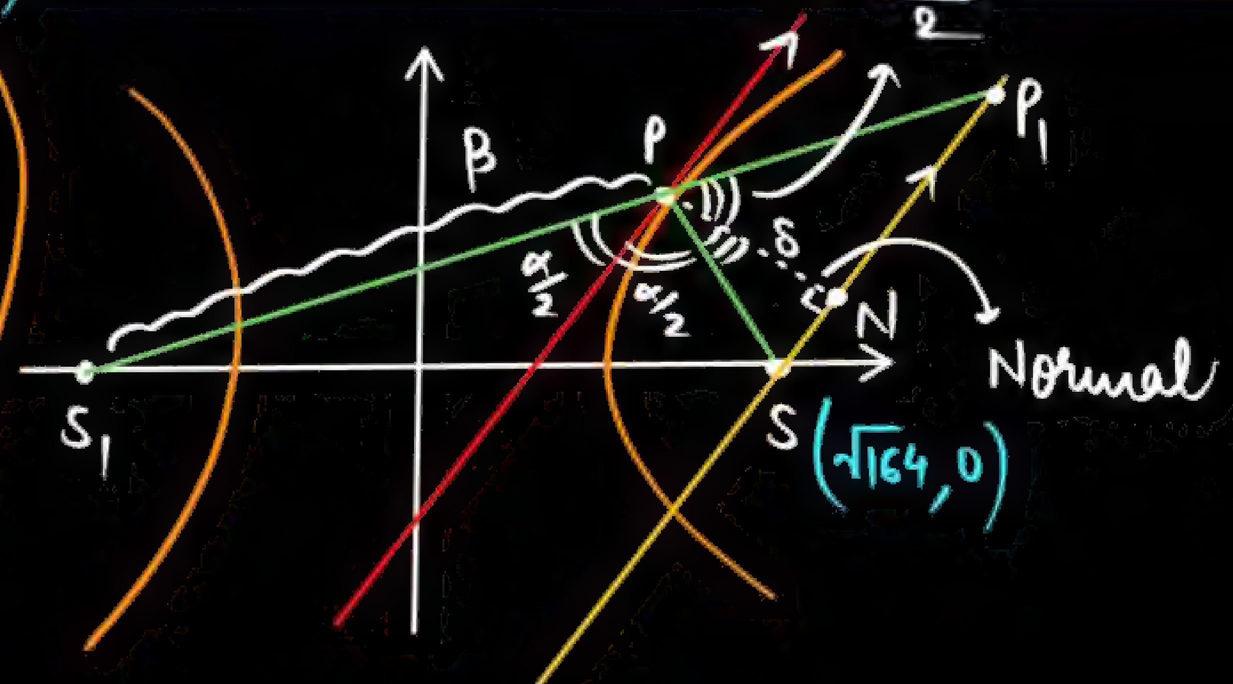
$e = \frac{\sqrt{164}}{10}$

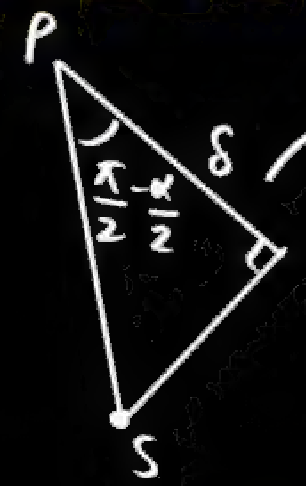
$\beta \sin\frac{\alpha}{2} - \delta \Leftarrow \beta - \frac{\delta}{\sin\frac{\alpha}{2}} = 2(10)$

$= 20\sin\frac{\alpha}{2}$



$PS_1 - PS = 2a$





$\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \frac{\delta}{PS}$

$\sin\frac{\alpha}{2} = \frac{\delta}{PS}$

Q. The number of points of intersection of $|z-(4+3i)|=2$ and $|z|+|z-4|=6$, $z \in C$ is:

[JEE Mains-2022]

H.W.

A. 0

B. 1

C. 2

D. 3

TODAY's HOMEWORK

MODULE

HYPERBOLA

Exercise – IV (PYQ) – COMPLETE

THE- END OF

COORDINATE GEOMETRY.

THANK YOU

to all future IITians